Political Equilibria with Social Security *

Michele Boldrin
Universidad Carlos III,
Madrid and C.E.P.R.

Aldo Rustichini
CentER
Tilburg University

April 1999

*Corresponding author: Michele Boldrin, Departamento de Economía, Universidad Carlos III, 28903 Getafe–Madrid, Spain. Phone/fax: +34-91-624 9595/624 9313. Michele@coral.uc3m.es

Abstract

We model PAYG social security systems as the outcome of majority voting within a OLG model with production. When voting, individuals make two choices: pay the elderly their pensions or default, which amount to promise themselves next period. Under general circumstances, there exist equilibria where pensions are voted into existence and maintained. Our analysis uncovers two reasons for this. The traditional one relies on intergenerational trade and occurs at inefficient equilibria. A second reason relies on the monopoly power of the median voter. It occurs when a reduction in current saving induces a large enough increase in future return on capital to compensate for the negative effect of the tax. We characterize the steady state and dynamic properties of these equilibria and study their welfare properties.

Keywords: Overlapping Generations, Political Economy, Social Security Systems, Taxation.
JEL Classification numbers: C72, C78.

* Financial support from the Fundación Marc Rich, CICYT (SEC96-0738) and the Fundación BBV is acknowledged. We are grateful to an anonymous referee of this Review for most valuable criticisms and suggestions. Seminar participants at the American Economic Association 1995 Meetings, Bilbao, Cambridge (U.K.), C.E.M.F.I. (Madrid), Chicago, C.O.R.E. (Louvain-la-Neuve), L.S.E., N.B.E.R. 1995 Summer Institute, Northwestern, Pompeu Fabra and UCLA contributed important comments.

1 Introduction

We are aging, and this may be more than just our personal and unavoidable experience. Indeed around the world, and particularly in the most advanced countries, the average citizen is becoming older. A sharp increase in the population's proportion of elderly and retired individuals has occurred in the last twenty years. An even more drastic shift in the same direction will take place in the near future, should the current demographic trends continue. At the same time, the growth rate of labor productivity has began to decrease.

These two phenomena compound into a drastic drop in the growth rate of the total wage bill. This has caused the political debate to move from the previous concentration on expanding the social security system, to concerns about the future viability of the system itself. Substantial reforms have already been implemented in some countries (Chile, Argentina, Mexico, Sweden, Italy) or are being proposed and starting to be implemented elsewhere. It is very likely that in the next twenty years similar phenomena will spread around the globe and that we will be facing the necessity to introduce drastic modifications in the way in which our social security systems (SSS) are organized and run.

Evaluating the social and economic impact of these predictions requires more than an examination of the demographic assumptions from which they are derived. It also requires an understanding of the process of collective decision making through which social security systems are shaped and managed.

In the literature one can find a variety of different explanations for why social security systems have been introduced (*e.g.* Becker and Murphy (1988), Diamond (1977), Kotlikoff (1987), Lazear (1979), Merton (1983), Sala-i-Martin (1992) to name but just a few). They seem to concentrate around the idea that public pension systems are efficiency enhancing, either because of the overaccumulation of capital that would occur without them, or because they provide for efficient risk-sharing in presence of incomplete annuities markets and adverse selection, or because they are a way around the lack of efficient intergenerational credit markets, or because they help reducing

While we find many of these motivations compelling, we believe a clearer theoretical understanding of the nature of modern social security systems requires an explicit consideration of three simultaneous features. Public pension systems redistribute wealth intergenerationally; they are implemented sequentially by each generation; they are sustained by the consensus of the majority of the working-age population. This is particularly true for pay-as-you-go (PAYG) systems (as opposed to Fully Funded ones), which are currently the rule in almost all advanced countries.

We are far from being the first to claim that PAYG SSS redistribute income between generations. This was made clear by Edgar Browning a while ago with two important articles, Browning (1973, 1975). He also pointed out that, because of the potential for intergenerational redistribution, median voter models predict that public social security plans will tend to be larger than required by economic efficiency. A number of contributions clarifying and extending Browning's initial intuition have followed since (see Verbon (1987) for a relatively recent update).

The temporal credibility problem of a PAYG pension system in the context of an OLG model with majority voting rule has been analysed in Sjoblom (1985). He considers a stationary environment, without capital accumulation or production, and shows that in the one-shot game there is no Nash equilibrium with SSS. In the repeated game he shows that there is a subgame perfect Nash equilibrium at which each player may achieve a pre-selected level of utility. This is shown in a deterministic setting by *a priori* restricting the set of beliefs adopted by the players. A median voter model of the Social Security System is also in Boadway and Wildasin (1989).

More recently Esteban and Sákovics (1993) have looked at institutions that can transfer income or wealth over time and across generations in the context of an OLG model. They model the creation of "institutions" by means of a fixed cost and consider both non-cooperative and cooperative games among generations. The basic intuition is similar to the one we use here, that is to say that "trust" can be built over time and maintained in order to achieve preferred outcomes. On the other hand they do not model the voting mechanism explicitly and are not interested in the dynamic interactions between the transfer system and the process of capital accumulation. Their analysis is nevertheless relevant to our own investigation as they characterize the efficiency properties of such institutions and the increase in efficiency achievable as set-up costs are reduced. In our model, set-up costs are set to zero.

At least four important aspects of the problem seem to have remained in the background of the previous literature: a) the set of conditions under which a PAYG social security scheme remains viable in an intertemporal context, when agents are selfish and are called to vote upon it in each period; b) the dynamic properties of such a system in the presence of an explicitly modeled accumulation of productive capital; c) the efficiency of the allocations that are achieved once the SSS is in place, particularly in those environments where intervention by a benevolent planner could be Pareto improving; d) the behavior of a PAYG SSS scheme in an environment in which it is known that, at some future date, the scheme will be dismissed and some generation which contributed will go away empty-handed. The latter is particularly relevant today: in front of the reasonable expectation that existing SSSs will become not viable in a few decades, should we expect a sudden and dramatic collapse immediately or should we instead prepare for a smooth phase out? Section **??** provides an explicit answer for this question (curious readers may jump ahead).

We show that a PAYG public pension plan is a subgame perfect equilibrium of a majority voting game in an OLG model with production and capital accumulation when the growth rate of total labor productivity and the initial stock of capital satisfy a certain set of restrictions. We characterize the properties of such systems for the general case and then give a detailed characterization of our model economy for selected functional forms of the utility function, the production functions and the stochastic process of the growth rate of population and labor productivity. For reasons of analytical simplicity we restrict our analysis to the case in which the uncertainty about the growth rate of the population is not distinguishable from that affecting the growth rate of labor productivity, but it would be relatively straightforward to duplicate all our results in the presence of random technology shocks affecting the size of the labor force and the marginal productivity of labor separately.

The traditional intuition for why a SSS may be desirable is that the young take advantage of the system of intergenerational transfers by exploiting the fact that the growth rate of the total wage bill is higher than the return on capital investment. In these circumstances it is advantageous to "invest" in the labor of the future generation, rather than in the future stock of capital, as long as the future generation also finds it profitable to "invest' in the labor productivity of the generation coming after it, and so on. This "efficiency inducing" motive is present also here, but it is not the only, nor the most important reason why the young generation may find it advantageous to support a PAYG system.

A second, less often noticed, motive is that by taxing labor income in order to finance current pension payments, the working generation directly affects the current saving's level and, through it, the total return from profits it will earn next period. When the return per unit capital is more than unitary elastic, a reduction in saving increases the total return more than proportionally, thereby increasing total lifetime income of the currently young generation. In other words, while the single member of the working generation must, when saving, act as a price-taker and accept whatever rate of return the market offers, he becomes a "collective monopolist" in the voting game. This implies that, for the working generation, a SSS may be advantageous, even when the growth rate of the total wage bill is lower than the rate of return on capital. This aspect of our model leads further support to the view, *e.g.* Feldstein (1987), that PAYG pension systems depress private saving. Indeed our analysis suggests that PAYG pension systems may even be established with the *intentional purpose* of inefficiently depressing private saving.

The examples of model economies we consider in the paper, are meant to play out each one of these motives. The economy we consider in the Example 1 of Sections **??**, **??**, **??** (linear utility and Cobb-Douglas production) stresses the "collective monopolist" aspect of the problem. The median voter only cares about total lifetime income and, therefore, if social security taxes turn out helpful to increase total capital income more than they decrease net labor income, he will be happy to adopt them, even if not needed on pure efficiency grounds. On the other hand, the model economy of section **??** (with log utility and linear production) concentrates on the "intergenerational trade" aspect of the problem. Quite interestingly we can show that, even when only the "efficiency inducing motive" is present, the sequential voting mechanism may not necessarily lead to an efficient outcome. A PAYG system gets introduced, but the working generation's pursuit of utility maximization sometime leads to too much and sometime to too little social security. The same example sends a soothing message to those concerned with a sudden collapse of the SSS. According to our model the knowledge that, at some time in the future, the current system will be dismantled because unprofitable for the generations then alive, does not imply its immediate dismissal today. If social decision making follows the logic we have tried to formalize here, a relatively smooth phasing out period should instead be expected.

Mention should be made here of the parallel and independent contributions by Azariadis and Galasso (1997), Cooley and Soares (1998) and Galasso (1998) addressing the existence and viability of a PAYG SSS from a point of view which is very similar to ours. Indeed all these authors adopt the overlapping generations framework to show that there exist equilibria in which Social Security is implemented by means of a majority voting mechanism. The basic intuition is therefore the same we introduce here, but all these contributions restrict their attention only to the case in which a constant Social Security tax is the equilibrium outcome and do not discuss the dynamic evolution of the political equilibrium and its impact on the accumulation of the aggregate stock of capital. Also, with the exception of Galasso (1998), the contributions by Azariadis and Galasso (1997) and Cooley and Soares (1998) consider only in passing the issue which is instead central to our analysis, *i.e.* the dynamic sustainability of a PAYG system over time as the long-run dynamics change both the available stock of capital and the total wage bill, and the form in which a collapse of the Social Security System may occur in front of a decrease in the growth rate of total labor productivity. On the other hand, these authors concentrate more on quantifying the welfare gains/losses generated by a social security tax by means of an explicit parameterization and of numerical simulations of the model at its steady state position. Our exercise and the ones just mentioned, therefore, appear as complementary rather than substitute.

A number of interesting properties of the equilibria we describe are consistent with the basic features of existing social security arrangements. For a PAYG pension system to be an equilibrium, it must entail a windfall for the generation of old people alive at the time of its introduction: they receive a transfer even if they had not made a contribution. In general the equilibrium PAYG system is not efficient, in the sense that it is not identical to the one that would be implemented by a benevolent government maximizing the utility of the average generation. Furthermore the equilibrium level of pension payments is linked to the real wages and it tends to get larger as income per capita grows, therefore behaving as a superior good.

Finally, some remarks should be made about the robustness of our more important results. In Sections **??**, **??** and **??** we use specific examples to show that the PAYG system are usually inefficient and may be so for either too much or too little capital accumulation. This result is perfectly general: for non trivial, neoclassical utility and production functions, a SSS which is established according to the dynamic voting game we study will, in general, be inefficient and may be so for either of the two reasons just mentioned. The same is true for the result of section **??**: the knowledge that the SSS will be dismantled at some future (but uncertain) date $T > 0$ does not imply, in general, its immediate collapse at $t = 0$.

The rest of the paper is organized as follows. In the next section we introduce the basic economic model, and its competitive equilibrium. In section **??** we introduce a simple voting model and characterize the sub-game perfect equilibria of the underlying game. Then we introduce the full social security game, and the definition of equilibrium. In the following sections we study the basic properties of the equilibria of this game, and its welfare properties. In section **??** we study the steady state equilibria, in section **??** the dynamics of the equilibrium paths. Section **??** is devoted to the welfare analysis. In section **??** we introduce a stochastic population growth rate to address the issue of the stability of the pay-as-you-go system. Section **??** concludes.

2 The basic economy

The economy is an OLG model with production, agents living for two periods, with no labor endowment during the second one. The population growth rate is exogenous, and $(1+n_t)$ is the ratio between young and old individuals alive in period t. Wage at time t is w_t, and profit is π_t. The tax rate is τ_t, and taxes are paid only on labor income. So $w_t\tau_t$ is the per-capita contribution of young people to the pension system, and $(1+n_t)w_t\tau_t$ the per-capita benefits to an elderly.

The representative young agent in period t solves

$$\max_{c_t, c_{t+1}} \quad u(c_t) + \delta u(c_{t+1}) \tag{2.1}$$

subject to:

$$\begin{aligned} c_t + s_t &\leq w_t(1-\tau_t), \\ c_{t+1} &\leq s_t\pi_{t+1} + w_{t+1}(1+n_{t+1})\tau_{t+1}, \\ 0 &\leq s_t \end{aligned} \tag{2.2}$$

When $s_t > 0$ the following first order conditions for interior solutions define the optimal saving of an agent:

$$u'(w_t(1-\tau_t) - s_t) = \delta u'(s_t\pi_{t+1} + w_{t+1}(1+n_{t+1})\tau_{t+1})\pi_{t+1} \tag{2.3}$$

The production function from per-capita capital k is denoted by f. The equilibrium conditions on the production side are:

$$\begin{aligned} w_t &= w(k_t) = f(k_t) - k_t f'(k_t) \\ \pi_t &= \pi(k_t) = f'(k_t) \\ k_{t+1} &= s_t/(1+n_{t+1}) \end{aligned} \tag{2.4}$$

We write c_t^s to denote the consumption at time t of an agent born at time s. For a given sequence (n_t, τ_t) of population growth rates and tax rates, a *competitive equilibrium* is a sequence $(w_t, \pi_t, c_t^t, c_{t+1}^t, s_t, k_t)_{t=0}^{\infty}$ of wages, profits, and consumption pairs (c_t^t, c_{t+1}^t) of the agent born at t, such that

i. consumption and savings solve the maximization problem (**??**) subject to (**??**) above, and

ii. wages, profits and capital stock satisfy the three equilibrium conditions just given.

Since wages and profits are function of the capital in that period, we may think that the optimal savings of an agent are a function of the present and future capital, and of present and future tax rates. We state formally the definition of two important objects.

Definition 2.1

*i. The individual saving function $\hat{s}$ is the function of $(k_t, k_{t+1}, \tau_t, \tau_{t+1})$ that solves the maximization problem (**??**) of the agents, where wages and profits are given by (**??**);*

ii. The equilibrium saving function is the function s^* of $(k_t, \tau_t, \tau_{t+1})$ that solves:

$$s^*(k_t, \tau_t, \tau_{t+1})) \in \hat{s}\left(k_t, \frac{s^*(k_t, \tau_t, \tau_{t+1}))}{1+n_{t+1}}, \tau_t, \tau_{t+1}\right) \tag{2.5}$$

In (ii) above we write the inclusion, rather than equality, to cover those cases in which the solution of the individual saving problem is not unique, as in some of our examples. The equilibrium saving function is determined as a fixed point: its value is a solution of the optimization problem of the individual agents, when they take as given the future capital supplied

When $n_t = n$ and $\tau_t = \tau$ for every t, we define a *steady state equilibrium* as an equilibrium sequence of *constant* wages, profits, saving, capital and consumption pairs.

3 The economy with voting

Consider now the same framework as before but assume the social security tax τ_t is not a fixed sequence. It is instead chosen by the citizens through majority voting. In each period both generations vote on whether to have a Social Security System (SSS) and at what level it should be implemented.

More precisely assume the political decision making process operates in the following form. If at time t a SSS does not exist, voters will decide to either remain without one or introduce it. In the second case they have to decide what is the per-capita transfer (if any) to be paid to the currently old generation, $\tau_t w_t$, and simultaneously make a "proposal" as to the amount of resources, $(1+n_{t+1})\tau_{t+1}w_{t+1}$, a currently young individual will receive next period, when retired. If a SSS is already in place, citizens are asked if they like to disband it or not. Keeping it, entails paying the promised amount to the current elderly and setting a promise for the payment to the elderly of next period. After the political process is completed and taxes are set, agents make their consumption and saving decisions and, as in the previous section, a competitive equilibrium is determined for given values of τ_t.

Individuals are homogeneous within a given generation. Obviously, the elderly will always favor keeping the SSS as the latter is financed by taxes on current wage income. The young workers, instead, face an interesting trade off. They know that, if the SSS is approved at time t and maintained thereafter, they have the equilibrium value of the problem:

$$\max_{c_t, c_{t+1}} \quad u(c_t) + \delta u(c_{t+1}) \tag{3.1}$$

$$\text{subject to} : c_t + s_t \leq w_t - d_t$$

$$\text{and} : c_{t+1} \leq \pi_{t+1} s_t + b_{t+1}.$$

where d_t is given and b_{t+1} has to be chosen. If instead the SSS is introduced at time t but rejected during the following period, the budget constraints in (3.1) become

$$c_t + s_t \leq w_t - d_t; \qquad c_{t+1} \leq \pi_{t+1} s_t. \tag{3.2}$$

On the other hand, if the SSS is either not in place or has been cancelled at time t but is then introduced at time $t+1$, the constraints are

$$c_t + s_t \leq w_t; \qquad c_{t+1} \leq \pi_{t+1} s_t + b_{t+1}. \tag{3.3}$$

Finally, if the SSS is absent in both periods the budget constraints of problem (3.1) become:

$$c_t + s_t \leq w_t; \qquad c_{t+1} \leq \pi_{t+1} s_t. \tag{3.4}$$

The key point is that while the value associated to (3.3) dominates the one associated to (3.4) which in turn dominates the one given by (3.2) there is no ways in which, a-priori, one may rank the payoffs associated to (3.1) and (3.4). In general, their ranking will depend on the population growth rate n_t, on the present and future stocks of capital (k_t, k_{t+1}) and on the promised level of benefits for the currently retired, b_t.

Under certain circumstances, the young agents can maximize utility by introducing or maintaining a SSS, and changing their saving-consumption decisions accordingly. This requires finding a τ_{t+1} that more than compensates for the payment of τ_t today and which (for identical reasons) may be acceptable to young voters next period.

An equilibrium with Social Security is now described by a sequence of capital stocks plus a sequence of $\{\tau_t\}_{t=0}^{\infty}$ such that, at the equilibrium of the economy where the τ's are taken as given, the value for the young generation in period t is at least as large as in the equilibrium without Social Security, for all t's. A formal definition is given in section 3.2.

Introducing a SSS entails two logical steps. First, one wants to find an infinite sequence of transfers (b_t, d_t) that, relative to the endowment position, improves the utility of all future median voters. Secondly, one needs to verify that each median voter has the incentive to approve the transfer b_t given that he expects to receive d_{t+1} next period. When the first requirement is satisfied the second is the straightforward equilibrium implication of a simple game with an infinite number of players. In the next subsection we will illustrate it only to focus the reader's intuition before moving on to the more complicated endogenous determination of the sequence (b_t, d_t) in an environment with endogenous capital accumulation.

Games more or less similar to the one introduced in the next subsection have been studied extensively in the literature, beginning with Hammond (1975) and then by Salant (1991), Kandori (1992), and Esteban-Sákovics (1993). In all of these cases, however, the game is of the repeated type, while in the complete game we consider later (section 3.2) the state variable (capital stock) is affected by the action and affects the payoff of each generation, thereby playing a key role in determining the

3.1 A Simple Game

Consider the following game. There are countably many players who move sequentially. At time t the designated player can choose to give to the previous player a predetermined amount d_t. If such payment is actually made, the player receiving the transfer will in fact be paid the amount $D_t \geq d_t$. The action of each player is perfectly observable by all those following him. As the amounts d_t and D_t are given, the actions available to each player are just

$$a_t \in \{1, 0\}$$

where 1 stands for *yes* and 0 stands for *no*. A history h of the game at time t, when it is player's t turn to move, is:

$$h = (a_1, a_2, \ldots, a_{t-1})$$

and we call H^t the set of such histories. A strategy for player t is a map

$$\sigma_t : (a_1, a_2, \ldots, a_{t-1}) \mapsto \{1, 0\}.$$

The payoff for player t depends on his action and the action of the next player, and is represented by a function $V : \{0,1\} \times \{0,1\} \mapsto \Re$. We assume that

$$V(0,1) > V(1,1) > V(0,0) > V(1,0). \tag{3.5}$$

The first and the last inequality follow if V is increasing in the transfer received and decreasing in the transfer paid. The second inequality requires that there is a net gain from paying a transfer and receiving one: if it does not hold, the system of transfers has no reason to exist. The subgame perfect equilibria (SPE) of this game are easy to describe.

Proposition 3.1 *A sequence of strategies $(\sigma_t)_{t=0}^{\infty}$ is a subgame perfect equilibrium if and only if for every $t \geq 0$ and every history $h \in H^t$:*

$$\text{if } \sigma_t(h) = 0, \ \text{then } \sigma_{t+1}(h, 0) = 1 \text{ or } \sigma_{t+1}(h, 1) = 0 \tag{3.6}$$

and

$$\text{if } \sigma_t(h) = 1, \ \text{then } \sigma_{t+1}(h, 0) = 0 \text{ and } \sigma_{t+1}(h, 1) = 1 \tag{3.7}$$

In fact the sequence of strategies is a subgame perfect equilibrium if and only, for every t and every history $h \in H^t$

$$V\Big(\sigma_t(h), \sigma_{t+1}(h, \sigma_t(h))\Big) > V\Big(\sigma_t(h)', \sigma_{t+1}(h, \sigma_t(h)')\Big) \tag{3.8}$$

where $\sigma_t(h)'$ is the action opposite to $\sigma_t(h)$. Now consider two cases separately. When $\sigma_t(h) = 0$, then (3.8) holds if and only if $V(0, \sigma_{t+1}(h, 0)) > V(1, \sigma_{t+1}(h, 1))$, and from the condition on V this is the case if and only if (**??**) holds. When $\sigma_t(h) = 1$, then (3.8) holds if and only if $V(1, \sigma_{t+1}(h, 1)) > V(0, \sigma_{t+1}(h, 0))$, and from the condition on V, this is the case if and only if (**??**) holds.

Now that we know the SPE strategy profiles we can characterize the SPE outcomes.

Proposition 3.2 *For any SPE $(\sigma_t)_{t=0}^{\infty}$, for every t and every history $h \in H^t$, if $\sigma_t(h) = 1$, then $\sigma_{t+i}(h, 1, \ldots, 1) = 1$, for every $i \geq 1$. So all SPE outcomes $(a_1^*, a_2^*, \ldots)$ of the game, are of the form:*

$$(0, 0, \ldots, 0, 1, 1, \ldots). \tag{3.9}$$

Conversely, for any T there exists a SPE outcome of this form with the first 1 at time T.

The first statement follows by iterating the second implication of condition (**??**). It follows that for every SPE outcome, if $a_t^* = 0$ then $a_s^* = 0$ for every $s < t$; therefore all SPE equilibrium outcomes have the form described in equation (3.9), where the 1 may never appear.

For the last statement, it is enough to define the equilibrium strategies as follows. Let

$$\sigma_t(h) = 0$$

for every $h \in H^t$, if $t < T$, and

$$\sigma_T(h) = 1,$$

for every $h \in H^T$,

$$\sigma_t(h) = \min\{a_T, \ldots, a_{t-1}\}$$

for every $t \geq T + 1, h \in H^t$. It is easy to verify that these strategies satisfy the conditions of the first proposition.

This simple game has shown a few important ideas. First, a social security system can be sustained at equilibrium. Second, the system can *arise* at equilibrium: that is, can become active after a number of periods in which it was not. Finally the SSS, once established, is never dismantled.

The first two results may be appealing; but the last conclusion sounds counterfactual. The intuitive reason for the third result is clear. If the system is going to be dismantled in some deterministic time in the future, the last generation which is to pay under this equilibrium will, in fact, refuse to do so, and the candidate equilibrium will unravel. We dedicate section 7 to the study of a model where the time in which the SS is dismantled is not deterministic even if it is known that, with probability one, the system will be dismantled at some finite future time. We show there that, in these circumstances, the system may be introduced, and then eventually abandoned *at equilibrium*.

Let us see how the intuition developed in the simple game can be transferred to a complete model of Social Security.

3.2 The Social Security Game

In the simple game that we have analyzed so far the amount of the transfers d_t and D_t was exogenous. To obtain a complete model of the social security game we will now express these quantities as tax receipts from labor income, on the one hand, and social security transfers on the other and then derive a political decision rule through which their amounts may get determined. This will require a richer description of a history of the game.

The action in our game is the decision of whether to pay the transfers to the previous generation or not, and the setting of the tax rate that will be paid in the next period. This is the strategic component, and its outcome is determined by the result of the voting mechanism.

Once the tax rate is given, quantities and prices are determined at competitive equilibrium. Now we go to the formal definitions.

A history h_{t-1} at time t is a sequence:

$$h_{t-1} = (a_1, \tau_1, a_2, \tau_2, \ldots, a_{t-1}, \tau_{t_1})$$

where for each $s \in (1, \ldots, t-1)$, $(a_s, \tau_s) \in \{1, 0\} \times [0, 1]$. Take now any infinite history $h \equiv (a_1, \tau_1, \ldots, a_t, \tau_t, \ldots)$. For any pair of adjacent actions $[(a_{t-1}, \tau_{t-1}), (a_t, \tau_t)]$ in the history the *effective tax rate* ($\tilde{\tau}_t$) at time t is defined to be $\tilde{\tau}_t = \tau_{t-1}$ if $a_t = 1$, and equal to 0 otherwise.

The competitive equilibrium of the economy given the sequence $(n_t, \tilde{\tau}_t)$ is well defined, and so is the lifetime utility of each agent. We have most of the elements necessary for a well defined game: action sets and payoffs; if we take each generation of agents as a single player, (that we call for convenience the *generation player*), we have a completely specified game. Formally we say:

Definition 3.3 *The* **generation player game** *of the economy is the extensive form game where:*

i. players are indexed by $t = (1, 2, \ldots)$*;*

ii. the action set of each player is $\{\mathcal{Y}, \mathcal{N}\} \times [0, 1]$*;*

iii. for every history $(a_1, \tau_1, a_2, \tau_2, \ldots, a_t, \tau_t, \ldots)$ *of actions, the payoff to each player is equal to the lifetime utility of the representative agent born at time* t *in the competitive equilibrium of the economy with* $(n_t, \tilde{\tau}_t), (n_{t+1}, \tilde{\tau}_{t+1})$.

Now that we have defined the game, we may proceed with the definition of the equilibrium.

Definition 3.4 *For a given sequence* (n_t) *of population growth rates, a* **political equilibrium** *is a sequence* $\{\tilde{\tau}_t, w_t, \pi_t, c_t^t, c_{t+1}^t, s_t, k_t\}_{t=0}^{\infty}$ *such that:*

(1) the sequence $\{w_t, \pi_t, c_t^t, c_{t+1}^t\}_{t=0}^{\infty}$ *is a competitive equilibrium given* $\{n_t, \tilde{\tau}_t\}_{t=0}^{\infty}$*;*

(2) there exists a sequence of strategies $\{\sigma_t\}_{t=0}^{\infty}$ *for the generation player game which is a subgame perfect equilibrium, and such that* $\{\tilde{\tau}_t\}_{t=0}^{\infty}$ *is the sequence of effective tax rates associated with the equilibrium history.*

In the next sections we study the equilibria that we have just defined. We begin by studying carefully a simple, but revealing example.

3.3 Example 1: Linear utility and CD production

Let us consider the equilibria in a simple economy with linear utility and Cobb Douglas production function: $u(c) = c$ and $f(k) = k^{\alpha}$.

We first solve the optimization problem of the agent. For a given pair of tax rates (τ_t, τ_{t+1}), a current and a future capital stock (k_t, k_{t+1}) the optimal saving of a representative agent solves

$$\max_{s \geq 0} \;(1 - \tau_t) w_t - s + \delta[\pi_{t+1} \cdot s + (1 + n) w_{t+1} \tau_{t+1}] \tag{3.10}$$

The solution to this problem is $(1 - \tau_t) w_t$, or the interval $[0, (1 - \tau_t) w_t]$, or 0, if $-1 + \delta\pi_{t+1}$ is positive, zero, or negative respectively. This is the optimal saving function. Note that $\delta\pi(k_{t+1}) \geq 1$ if and only if $k_{t+1} \leq (\alpha\delta)^{1/(1-\alpha)}$.

Next we determine the equilibrium saving function (recall the definition **??**). Consider the value of the maximum saving: $(1 - \tau_t) w(k_t)$. When the value of k_{t+1} is such that $\delta\pi(k_{t+1}) \geq 1$, then the maximum saving is the equilibrium saving. When the value of the capital stock given by the maximum savings is higher, then the supply of savings reduces, until the level of next period capital stock is exactly $(\alpha\delta)^{1/(1-\alpha)}$. To summarize, the equilibrium saving function is:

$$s^*(k_t, \tau_t, \tau_{t+1}) = \min\Big\{(1 - \tau_t)(1 - \alpha) k_t^{\alpha},\; (1 + n)(\alpha\delta)^{1/(1-\alpha)}\Big\} \tag{3.11}$$

which is independent of τ_{t+1}. The evolution of the per-capita stock of capital is

$$k_{t+1} = \min\left\{ \frac{(1-\alpha)(1-\tau_t)k_t^\alpha}{(1+n_t)},\ (\alpha\delta)^{1/(1-\alpha)} \right\} \tag{3.12}$$

The value of social security

The value of keeping the SSS as a function of k_t, k_{t+1}, the current tax rate τ_t and of the proposed tax rate τ_{t+1} is obtained by substitution of the equilibrium saving function $s^*(k_t, \tau_t, \tau_{t+1})$ into the utility function of the agent. The result is

$$\begin{aligned} V(k_t, \tau_t, \tau_{t+1}) &= (1-\tau_t)(1-\alpha)k_t^\alpha - \\ &- s^*(k_t, \tau_t) + \delta(1+n)^{1-\alpha} s^*(k_t, \tau_t)^\alpha(\alpha + (1-\alpha)\tau_{t+1}). \end{aligned} \tag{3.13}$$

On the other hand, the value for the representative member of the young generation of turning down the SSS and going alone with tax rates $\tau_t = \tau_{t+1} = 0$ is equal to $V(k_t, 0, 0)$, denoted by $v(k_t)$, and given by:

$$v(k_t) = (1-\alpha)k_t^\alpha - s^*(k_t, 0) + \delta\alpha(1+n)^{1-\alpha} s^*(k_t, 0)^\alpha \tag{3.14}$$

4 Steady State Equilibria

Some equilibrium outcomes are particularly simple to study; as usual, steady states are first among them. We define a *political equilibrium steady state* to be the political equilibrium where all quantities are constant in time; in the following $k(\tau)$ will denote the steady state value of the *per capita* capital stock, in a competitive equilibrium where the sequence of tax rates is fixed at τ. Also $V(k(\tau), \tau)$ denotes the lifetime value of utility for the representative generation, at the steady state value of the capital stock associated to a stationary tax rate $\tau > 0$.

Suppose now that, with $k = k(\tau)$, the social security system is voted down. The sequence of tax rates is then set to zero, and the sequence of values of capital stocks in the competitive equilibrium will in general diverge from $k(\tau)$. The generation of agents born in that period will achieve a level of utility different from $V(k(\tau), \tau)$, and dependent only on the value of $k(\tau)$; let $v(k(\tau))$ denote this value. Note that clearly

$$V(k(0), 0) = v(k(0)).$$

Which values of taxes and capital stock can be supported as steady state political equilibria? Clearly a necessary condition is that

$$V(k(\tau), \tau) \geq v(k(\tau))$$

since otherwise the social security system would be immediately voted down. But it is easy to see that the condition is also sufficient:

Proposition 4.1 *The steady state value $k(\tau)$ can be supported as the outcome of a political equilibrium if and only if $V(k(\tau),\tau) \geq v(k(\tau))$.*

Proof. The sequence of strategies $\{\sigma_t\}_{t=0}^{\infty}$ where, for every history h_t, $\sigma_t(h_t) = (\mathcal{Y},\tau)$ if for every $s < t$, $(a_s, \tau_s^h) = (\mathcal{Y},\tau)$, and equal to $(\mathcal{N},\tau)$ otherwise, satisfies our definition of political equilibrium, and supports the steady state. ■

4.1 Example 1 (continue)

Consider again the example 1, where $u(c) = c$ and $f(k) = k^\alpha$. If the tax rate and population growth rates are constant, with $\tau_t = \tau$ and $n_t = n$, then equation (3.12) has a unique non zero stable steady state, denoted by $k(\tau)$ and given by:

$$k(\tau) = \left(\min\left\{\frac{(1-\alpha)(1-\tau)}{1+n},\ \alpha\delta\right\}\right)^{1/(1-\alpha)} \tag{4.1}$$

We now distinguish two cases, depending on where the minimum is achieved. The case in which $(1-\tau)(1-\alpha) > (1+n)\alpha\delta$ is the least interesting. The steady state value of capital is only dependent on its marginal productivity and the discount factor, and is independent of τ. The difference between the value of keeping the SSS and the value of dismantling it, at the steady state capital, is easily computed to have the same sign as $\delta(1+n)-1$. So a positive tax rate can be sustained if and only if any positive tax rate can be sustained, that is if and only if $\delta(1+n) \geq 1$.

When $(1-\tau)(1-\alpha) < (1+n)\alpha\delta$, the steady state capital is

$$k(\tau) = \left[\frac{(1-\alpha)(1-\tau)}{1+n}\right]^{1/(1-\alpha)}$$

and the equilibrium saving function is equal to the maximum saving, $(1-\tau)w(k(\tau))$. So we find from (3.13)

$$V(k(\tau),\tau) = \delta(1-\tau)^\alpha(1+n)^{1-\alpha}[\alpha + (1-\alpha)\tau]w(k(\tau))^\alpha \tag{4.2}$$

from (3.14)

$$v(k(\tau)) = \delta\alpha(1+n)^{1-\alpha}w(k(\tau))^\alpha \tag{4.3}$$

The difference between V and v has the same sign as $g(\tau) - \alpha$, where

$$g(\tau) \equiv (1-\tau)^\alpha[\alpha + \tau(1-\alpha)].$$

It is easy to see that:

i. $g(0) = \alpha$, $g(1) = 0$

ii. the function $\log g$ is concave.

So there exists a solution to $g(\tau) = 0$, call it τ_p, if and only if $g'(0) > 0$, and the solution is unique when it exists. But $g'(0)$ has the same sign as $1 - \alpha - \alpha^2$. We conclude:

Proposition 4.2

i. A SSS with constant tax rate τ such that $(1-\tau)(1-\alpha) > (1+n)\alpha\delta$ can be sustained at steady state if and only if $\delta(1+n) \geq 1$;

ii. A SSS with constant tax rate τ such that $(1-\tau)(1-\alpha) < (1+n)\alpha\delta$ can be sustained at steady state if and only if $\tau \in [0, \tau_p]$; in particular a steady state equilibrium with social security exists if and only if $1-\alpha-\alpha^2 \geq 0$.

4.2 Example 2: Log utility and linear production

Let $u(c) = \log(c)$, $f(k) = ak + b$, and $n_t = n$ for all t. This example is extremely simple and also somewhat paradoxical (for instance, one of the possible steady state capital stocks at the political equilibrium is equal to zero, and there is no interesting dynamics in the capital stock), but we introduce it for two reasons. First, it should help to clarify the intuition. And second, it is the basis for the stochastic model we introduce later (in section **??**).

In this model the marginal productivity of capital is constant and therefore independent of the level of saving. So the incentive for the young generation to manipulate the total return on capital through the social security tax does not play any role here. The only motivation behind the choice of establishing a SSS is the potential intergenerational gain to the young generation. Nevertheless, as the analysis in this section and in section **??** will show, the political equilibrium does not always implement an efficient allocation

Equilibrium paths

Note that:

$$w_t = b, \pi_t = a, \forall t \tag{4.4}$$

The optimal saving function is

$$s_t = \max\left\{ b\frac{a\delta(1-\tau_t) - (1+n)\tau_{t+1}}{a(1+\delta)}, 0 \right\} \tag{4.5}$$

Since the saving function does not depend on the current stock of capital, this is also the equilibrium saving function.

Since at optimal interior consumption paths $c_{t+1} = a\delta c_t$, the value to a generation, when savings are positive, is equal to $(1+\delta)\log c_t + \delta \log(a\delta)$. From (4.5) and $c_t = (1-\tau_t)b - s$ we conclude: if $s > 0$, then

$$\begin{aligned} V(k_t, \tau_t, \tau_{t+1}) = {} & (1+\delta)\log[a(1-\tau_t) + b(1+n)\tau_{t+1}] - \\ & -(1+\delta)\log[a(1+\delta)] + \delta\log(a\delta) \end{aligned} \tag{4.6}$$

When $\tau_{t+1} = 0$, and in particular when no SSS exists, the optimal savings are positive, so from (4.6) we get:

$$v(k_t) = (1+\delta)\log\left(\frac{b}{1+\delta}\right) + \delta\log(a\delta) \tag{4.7}$$

Finally, when optimal savings are zero, consumption is given by the budget constraint, and , if $s = 0$, then

$$V(k_t, \tau_t, \tau_{t+1}) = \log[b(1-\tau_t)] + \delta\log[b(1+n)\tau_{t+1}] \tag{4.8}$$

Sustainable taxes

In the appendix we prove that a fixed level of tax rates is sustainable if and only if the population growth rate is larger than the marginal productivity of capital:

Proposition 4.3 *A fixed tax rate τ can be sustained if and only if*

$$a < 1 + n. \tag{4.9}$$

If (4.9) *is not satisfied, then no SSS with positive tax rates can be sustained.*

When the marginal productivity of capital is too large, the SSS cannot be sustained. More precisely, the proof shows that the minimum tax rate which is necessary in each period to make the SSS viable is growing without bounds.

This is the basic idea behind the stochastic model we discuss in section **??**. When population growth is declining, we have an initial situation in which (4.9) is satisfied, but as time goes by that condition is eventually violated. So the SSS was initially sustainable, but eventually collapses. In section **??** we show that this can happen at equilibrium when the decline in population is stochastic, so no generation knows for sure that it will be the last one to pay.

5 The Dynamics of Political Equilibrium

We now move to the harder task of determining the equilibrium paths out of an initial condition on the capital stock that is not necessarily a steady state. Our aim is to determine how the capital accumulation evolves when the decision on tax rates is determined by a voting mechanism. Given the difficulty of providing a complete characterization for the general case we will proceed by means of the same example we have just introduced. Additional examples can be found in the working paper version of this article, Boldrin and Rustichini (1995).

Example 1 (continue)

To organize the analysis it is convenient to distinguish three separate regions in the capital stock-tax rate space $\Re_+ \times [0, 1]$:

$\mathbf{R1} \equiv \{(k,\tau) : (\alpha\delta)^{\frac{1}{1-\alpha}}(1+n) \leq (1-\tau)w(k)\}$;

$\mathbf{R2} \equiv \{(k,\tau) : (1-\tau)w(k) \leq (\alpha\delta)^{\frac{1}{1-\alpha}}(1+n) \leq w(k)\}$;

$\mathbf{R3} \equiv \{(k,\tau) : w(k) \leq (\alpha\delta)^{\frac{1}{1-\alpha}}(1+n)\}$

Note that the boundary between the second and the third region is a line $\{(k^3,\tau) : \tau \in [0,1]\}$. The equilibrium saving in the regions **R2** and **R3** is equal to $(1-\tau)w(k)$, and to $(\alpha\delta)^{\frac{1}{1-\alpha}}(1+n)$ in **R1**. From the form of the equilibrium saving function we have that two cases are possible for the steady state when the tax rate is identically zero: it is either equal to $(\frac{1-\alpha}{1+n})^{\frac{1}{1-\alpha}}$ or it is equal to $(\alpha\delta)^{\frac{1}{1-\alpha}}$. We concentrate on the first, more interesting, case and assume in the rest that:

$$\frac{1-\alpha}{\alpha} \leq \delta(1+n).$$

Let us immediately note three implications of this assumption: First, $(\alpha\delta)^{\frac{1}{1-\alpha}}(1+n) \leq k^3$. Second, k^3 is larger than the steady state of the economy with zero taxes, and therefore also larger than the steady state of the economy with any tax rate. Finally from the form of the equilibrium saving function we have that for any initial condition and any sequence of tax rates, $k_t \leq k^3$ for any t after the initial period: in other words, the third region is invariant for any competitive equilibrium path beginning within it.

In the third region the equilibrium saving function is equal to $(1-\tau)w(k_t)$, and it is easily found that the inequality $V(k_t,\tau_t,\tau_{t+1}) \geq v(k_t)$ is equivalent to

$$\alpha + (1-\alpha)\tau_{t+1} \geq \alpha(1-\tau_t)^{-\alpha} \tag{5.1}$$

An interesting property of (5.1), which will be used repeatedly in the future, is that it does not involve the stock of capital. Rewriting (5.1) in explicit form gives

$$\tau_{t+1} \geq \frac{\alpha}{1-\alpha}\Big[\frac{1}{(1-\tau_t)^{\alpha}} - 1\Big] = \phi(\tau_t)$$

If the restriction $0 \leq \alpha < (\sqrt{5}-1)/2 = \alpha^*$ is satisfied the convex function ϕ starts at the origin and has a unique interior fixed point τ_p which is unstable under repeated iterations. For larger values of α only the degenerate equilibrium without SSS exists.

To complete the analysis of the dynamics of the political equilibria, proceed as follows. Let $0 \leq \alpha < \alpha^*$ For any τ_t larger than τ_p the next period transfers necessary to make the social security system acceptable to the next generation are (recall that the third region is invariant) equal to $\phi(\tau_t)$ or larger, and iteration of this argument leads to a tax rate larger than 1 in finite time, hence the system cannot be supported. On the other hand as long as the tax rate remains in the region $[0,\tau_p]$ the social security system can be supported. In fact we can completely characterize the set of equilibrium paths. Define the correspondence

$$\Phi(\tau) \equiv [\phi(\tau),\tau_p]$$

and then:

Proposition 5.1 *A sequence (k_t, τ_t) is an equilibrium outcome, if and only if it is a solution of the system of difference inclusions:*

$$k_{t+1} = \frac{s^*(k_t, \tau_t)}{1+n}; \quad \tau_{t+1} \in \Phi(\tau_t),$$

for every $t \geq 1$.

The proof of this statement is immediate.

We have proved earlier that the third region is invariant, and therefore the comparison between the two values V and v only depends on the tax rates, as formulated in equation (5.1) above. Equilibrium strategies are easy to define in these circumstances and we may concentrate on a few, more interesting equilibria.

In the first equilibrium the tax rate is immediately set equal to the highest possible value compatible with the existence of a SSS. When the initial capital stock is less or equal than k^3, this rate is equal to τ_p. Then the equilibrium path has the form (k_t, τ_p), where the sequence k_t converges to the steady state k_{τ_p}. When the initial capital stock is higher than k^3, the highest possible tax rate turns out to be lower than τ_p. In fact there is a curve in the capital-tax rate space, where the tax rate is decreasing with the capital stock, that describes the equilibrium tax rate. Details are provided just below, but first let us stress the intuition: when the capital stock is high the value of the alternative of dropping the social security system is too high, and the generation who introduces the social security system has to accept lower transfers to make the system viable. After one period, however, the equilibrium path enters the third region, and the time sequence is by now familiar.

To see how the intuition operates, recall that under our assumption on the parameters, from any initial capital stock larger than k^3 the competitive equilibrium sequence, irrespective of the tax rate, has value less than k^3 after one period; so we are looking for the highest tax rate that makes the function V larger than v, conditional on the next period tax rate being equal to τ_p. Let us now begin to describe the first region. Here the inequality $V(k, \tau, \tau_p) \geq v(k)$ is found to be equivalent to

$$\delta(1+n)(\alpha\delta)^{\frac{\alpha}{1-\alpha}}\tau_p \geq \tau k^{\alpha}.$$

The equality determines a function from τ to k; it is immediate that this function is decreasing in τ. The same remarks we just made hold in the case of the second region. Here the inequality between V and v, again conditional on the next period tax rate being equal to τ_p, is equivalent to:

$$\delta(1+n)[\alpha + (1-\alpha)\tau_p] \geq (1-\alpha)^{1-\alpha} k^{\alpha(1-\alpha)} (1-\tau)^{-\alpha}.$$

Calculus shows that the equality determines a function from tax rate to capital stock, decreasing, and with value at τ_p equal to k^3, as it should.

In the second equilibrium we consider the sequence of tax rates defined by:

$$\tau_{t+1} = \phi(\tau_t).$$

For simplicity we only consider the case $k_0 \leq k^3$. The tax rate converges to zero, and the capital stock converges to the steady state value $\left(\frac{1-\alpha}{1+n}\right)^{\frac{1}{1-\alpha}}$ of the economy with zero transfers. This equilibrium corresponds to the slow disappearance of the social security system.

6 Welfare Analysis

6.1 The Planner's Choice at Steady State

In the tradition of Diamond (1965) and Samuelson (1975) we consider the tax rates which are efficient at steady state. To be precise we define an *efficient steady state tax rate* as the tax rate which gives, at the associated value of steady state capital stock, the highest per capita utility to each generation. Our interest here is in showing that the steady state tax rate generated in a political equilibrium needs not satisfy any efficiency requisite. This should be intuitively clear from our discussion of the incentives the median voter faces in choosing period t's tax rate.

Comparing the efficient and the political steady state tax rates for the example we have considered along the paper, should be enough to drive home this simple point.

Example 1 (continue)

We have already analyzed the steady state political equilibria of $u(c) = c$, and $f(k) = k^\alpha$ in section 4. In the case in which taxes are chosen by a planner we have:

Proposition 6.1 *The efficient steady state tax rate is:*

$$\tau_e = \frac{1-2\alpha}{1-\alpha}$$

with corresponding steady state capital stock

$$k = \left(\frac{\alpha}{1+n}\right)^{1/(1-\alpha)}$$

if $\alpha \leq \frac{1}{2}$ *and* $\delta(1+n) > 1$. *The efficient steady state tax rate is equal to* 0 *otherwise, with steady state capital stock:*

$$k = \left(\frac{1-\alpha}{1+n}\right)^{1/(1-\alpha)}$$

The proof of this proposition can be found in the appendix. We return to this example in section **??**, to analyze the dynamic aspects of the issue.

6.2 The Dynamic of Taxes and Capital under a Planner

One can give an explanation for the inefficiency of the competitive equilibrium without social security by observing that the rates of return of the voluntary savings, s_t, and of the forced saving which would be achieved through the SSS, $\tau_t w_t$, are not equalized at the competitive equilibrium allocation. In fact, a necessary condition for efficiency is precisely that these two rates are equalized. This condition is not sufficient, as one can see from our example: there may be more than one steady state in which the condition is satisfied. This poses the next question.

From a normative point of view, we may consider a social planner who, in the initial period when a capital stock k_0 is given, chooses the initial tax rate arbitrarily. The tax rates for the following periods are then determined in order to equalize (in each period) the rate of return on capital to the rate of return implicit in the SSS. Under this interpretation of the benevolent planner's behavior, it is important to characterize the long run properties of the competitive equilibrium paths that are determined in this way; in particular, if the efficient steady states are stable. They will turn out not to be. In our framework the equality of rates requires

$$\frac{(1+n_{t+1})w_{t+1}\tau_{t+1}}{\tau_t w_t} = \pi_{t+1} \tag{6.1}$$

whenever the tax rates τ_t, τ_{t+1} are different from zero. At a steady state with positive tax transfers, this is the *Golden Rule*.

Replacing the factor market equilibrium conditions in (6.1) and adding the capital accumulation equation yields a two-dimensional dynamical system in implicit form

$$k_{t+1} = \frac{s[w(k_t), \pi(k_{t+1}), \tau_t, \tau_{t+1}]}{(1+n_{t+1})} \tag{6.2i}$$

$$\tau_{t+1} = \frac{\tau_t \cdot w(k_t) \cdot \pi(k_{t+1})}{(1+n_{t+1})w(k_{t+1})} \tag{6.2ii}$$

which applies for all quadruples $\{(k_t, \tau_t), (k_{t+1}, \tau_{t+1})\}$, as long as the optimal solution is interior. Fixing a constant growth rate of the population $n_t = n$ one can compute the unique steady state of (6.2) with positive taxes. As long as $k^* \geq 0$, this is given by the pair (k^*, τ^*) satisfying:

$$k^* = g^{-1}(1+n), \qquad (g \equiv f')$$

$$u'(w(k^*)(1-\tau^*) - k^*(1+n)) \geq \delta u'(k^*(1+n)f'(k^*) + w(k^*)(1+n)\tau^*)f'(k^*)$$

This is, obviously, the steady state at which the rate of return on capital $f'(k^*) - 1$ equals the rate of return on social security taxes, which also equals the rate of growth of the population n. It is immediate to verify that a low population growth rate induces a higher steady state level of per-capita capital stock $k^*(n)$. On the other hand, the effect of variations in n on the efficient size of the social security transfer is ambiguous.

Since the equation that gives equality of rates of return is satisfied for any value of capital when the tax rate is zero, there may be other steady states, of the form $(k^*, 0)$. For future comparison we characterize the behavior of the system for our usual pair of utility and production functions.

Example 1 (Continue)

We have learned earlier that the saving function is only a function of (k_t, τ_t):

$$s^*(k_t, \tau_t) = \min\{(1 - \tau_t)(1 - \alpha)k_t^\alpha,\ (1 + n_{t+1})(\alpha\delta)^{1/(1-\alpha)}\}$$

Together with the condition that equalizes the return on private saving to the return on social security payments, this yields a dynamical system for (τ_t, k_t)

$$k_{t+1} = \min\left\{\frac{(1 - \tau_t)(1 - \alpha)k_t^\alpha}{(1 + n)}, (\alpha\delta)^{1/(1-\alpha)}\right\}$$

$$\tau_{t+1} = \max\left\{\frac{\alpha\tau_t}{(1 - \alpha)(1 - \tau_t)}, \frac{\alpha\tau_t k_t^\alpha}{(1 + n)(\alpha\delta)^{1/(1-\alpha)}}\right\}$$

Setting aside the hairline case in which $\delta(1 + n) = 1$, and in accordance with the steady state analysis of subsection 6.1, we consider separately the two cases $\alpha < (\geq)1/2$.

When $\alpha < 1/2$ the dynamical system has two stationary states:

$$(k_1^*, \tau_1^*) = \left(\left(\frac{\alpha}{1 + n}\right)^{1/(1-\alpha)}, \frac{1 - 2\alpha}{1 - \alpha}\right)$$

and

$$(k_2^*, \tau_2^*) = \left(\left(\frac{1 - \alpha}{1 + n}\right)^{1/(1-\alpha)}, 0\right).$$

When $\delta(1 + n) > (1 - \alpha)/\alpha$ the largest of the two steady states is smaller than $(\alpha\delta)^{1/(1-\alpha)}$, and so eventually the behavior around the two stationary states is governed by the system

$$k_{t+1} = \frac{(1 - \tau_t)(1 - \alpha)k_t^\alpha}{(1 + n)}$$

$$\tau_{t+1} = \frac{\alpha\tau_t}{(1 - \alpha)(1 - \tau_t)}$$

The two eigenvalues defining the linear approximation near each of the steady states are respectively

$$\lambda_1(k_1^*, \tau_1^*) = \alpha, \quad \lambda_2(k_1^*, \tau_1^*) = \frac{1 - \alpha}{\alpha}$$

$$\lambda_2(k_2^*, \tau_2^*) = \alpha, \quad \lambda_2(k_2^*, \tau_2^*) = \frac{\alpha}{1 - \alpha}$$

Since $\alpha < 1/2$ the first steady state is a saddle and the second a sink. The stable manifold of (τ_1^*, k_1^*) is then a vertical line in the (τ, k) space in correspondence to the value $\tau = \tau_1^*$. As we have seen, for this value of the parameters the efficient steady state taxes are positive, and equal to τ_1^*. So *the efficient steady state is the unstable steady state*. For all initial conditions (τ_0, k_0) such that $\tau_0 < \tau_1^*$ the equilibrium path converges to the (inefficient) stationary state (τ_2^*, k_2^*) without SSS. When $\tau_0 = \tau_1^*$, it converges to the stationary state (τ_1^*, k_1^*). There is no equilibrium that begins with $\tau_0 > \tau_1^*$, since in that case the tax rate would become larger than 1 in a finite number of periods.

In the other case of $\alpha > 1/2$ only (k_2^*, τ_2^*) is an admissible steady state, hence the only efficient steady state, which is now unstable.

6.3 Welfare Properties of the Political Equilibria

Next we show, again by means of our example, that the political equilibria are typically inefficient, but may be so either because there is too much capital accumulation at equilibrium, or because there is too little capital accumulation.

Example 1 (continue)

The reason for the, possible, inefficiency of the competitive equilibrium without social security transfers is unambiguous and well known: without transfers there is an overaccumulation of capital at equilibrium. In the case we are considering, this is obvious from the equation determining the value of the steady state capital, which is decreasing in τ; so when it is efficient to have positive transfers it is so because the equilibrium capital will be reduced.

It is convenient to determine, first of all, the relative position of the efficient tax rate in the interval $[0, \tau_p]$. If we substitute the value $\frac{1-2\alpha}{1-\alpha}$ into the equation that determines τ_p we have that $\tau_p > \tau_e$ if and only if $\alpha < 1/2$.

Some easy calculus shows that τ_p is an increasing function of α, and that $\lim_{\alpha \to 0} \tau_p = 1$. There are therefore three completely different possibilities:

1. When $\alpha \in [\alpha^*, 1]$, the efficient level of tax rate is zero, but also the only tax rate that can be supported as a political equilibrium is zero. The productivity of capital is so high that it is never convenient for the median voter to manipulate the rate of return on capital by reducing the latter through the introduction of a SSS.

2. For all the values of $\alpha \in [1/2, \alpha^*]$ (and any δ), and for values of α and δ such that $\alpha < 1/2$ and $\delta(1+n) < 1$ there are many levels of tax rate that can be supported in a political equilibrium. In the same region the efficient level of tax rate is zero. So in this region any steady state political equilibrium with positive transfers is unambiguously inefficient, with a steady state level of capital stock which is too low. In this region the efficiency enhancing motive for introducing a

SSS is not present, nevertheless the gains from manipulating the total return on capital through the imposition of τ more than compensate for its costs.

3. The political equilibria are, typically, inefficient also in the rest of the parameter space. In this case, though, the political equilibrium tax rate may be either too high or too low (and the capital stock, therefore, too low or too high) with respect to the efficient level. To see this, notice that when $\alpha \leq 1/2$ and $\delta(1+n) > 1$ the maximum tax rate that can be supported at the political equilibrium is higher than the efficient rate and, consequently, the political equilibrium stock of capital is too low. Again the monopolistic motivation to lower saving, induces too high a tax rate. On the other hand, there are also political equilibria with very low tax rates, and too much physical capital. These equilibria are inefficient for the opposite reason.

7 Stochastic Population Growth

In the recent historical experience of many countries the immediate cause for the general alarm surrounding the social security system seems to be the long run fall in the growth rates of population and labor productivity. Since this is a long-term and quite foreseeable process, it is natural to ask how the SSS would react to it, and what predictions we can derive from the explanation that our model gives of such changes. The model that we present now will provide a first answer to these questions. As in the rest of the paper, and without any loss of generality, we restrict attention to changes in the growth rate of population, the extension to exogenous labor-augmenting technological progress being immediate.

7.1 Logarithmic utility, linear production and stochastic population growth

Recall our second example, logarithmic utility function and linear production function, introduced in Section 4.1. In this section we make the growth rate of population a stochastic process. More precisely we assume that there exists a sequence of growth rates $\{n(j)\}_{j=0}^{\infty}$ satisfying the restrictions $n(j+1) < n(j), \quad \text{for all } j$, and $\lim_{j\to\infty} n(j) = 0$, and a transition probability

$$\Pr\,(n_{t+1} = n(j); n_t = n(j)) = 1 - p,$$

$$\Pr\,(n_{t+1} = n(j+1); n_t = n(j)) = p;$$

for all j, with $0 < p < 1$. The definition of equilibrium with a stochastic growth rate of the population is very similar to the one we have adopted for the models with a deterministic dynamic of the population growth rate. We refer the reader to the appendix for a formal statement. We let

$$a > 1,$$

so that $a > 1 + n_t$, eventually almost surely.

The strategy profiles of the political equilibrium are defined as follows. Let n_t be $n(j)$, then

$$\sigma_t(h_t) = \Big(\mathcal{Y}, \tau(n(j)), \tau(n(j+1))\Big)$$

if for every $s < t$, $a_s = \Big(\mathcal{Y}, \tau(n(i)), \tau(n(i+1))\Big)$, where $n_s = n(i)$. On the other hand the strategy of the player t sets

$$\sigma_t(h_t) = \Big(\mathcal{N}, \tau(n(j)), \tau(n(j+1))\Big)$$

in all other cases. The mappings from $n(i)$ into $\tau(n(i))$ used here are defined explicitly in the proof of proposition **??**. The proof of the statement is a consequence of the characterization of the political equilibrium provided in the next two propositions.

Proposition 7.1 *If $a > 1$ then, for any political equilibrium of the generation player game, almost surely $\tau_t = 0$ for every $t \geq T$; that is the social security system is terminated almost surely.*

The proof is in the appendix.

Of course the proposition is interesting only if there are equilibria which give, as an outcome, tax rates which are not identically zero. This requires that, at least in the initial periods, the population growth rate is sufficiently high to make the return from the savings invested in the security system larger than the return from private investment. To avoid uninteresting situations we assume, in addition, that

$$a < (1-p)(1+n_0).$$

The linear production function rules out any reason for manipulating the aggregate saving rate via taxation. Our assumptions imply, therefore, that the only motive to sustain a SSS is the efficiency gain due to $(1-p)(1+n_t) > a$ for some $t > 0$, and that the latter motive also disappears in finite time.

We now prove that equilibria with positive tax rates and transfers in the initial periods exist, even if it is known that the system of social security will be eventually dismissed almost surely. As usual, the construction of an equilibrium rests on the comparison between the value of keeping the social security system, and the value of dropping it. In the model we are discussing the second value is a constant, v, that we computed in (4.7):

$$v = (1+\delta)\log\left(\frac{b}{1+\delta}\right) + \delta\log(a\delta).$$

First we prove a simple proposition which provides the constructive methods through which equilibria are found:

Proposition 7.2 *There is a sequence of tax rates* $\{\tau(j)\}_{j=0}^{\infty}$ *such that*

$$\max_{s\geq 0} \Big\{ \log[(1-\tau(j))b - s] + \delta(1-p)\log[as + (1+n(j))\tau(j)b] +$$

$$+\delta p \log[as + (1+n(j+1))\tau(j+1)b] \Big\} = v$$

for all j*'s, and* $\tau(j) > 0$ *for some* j.

The details of the proof are in the appendix. It is based on backward induction and encompasses the following steps:

i. First, for any $n(j)$ we determine the value of τ (call it $\underline{\tau}(j)$) which satisfies the following condition: "If $\underline{\tau}(j)$ is to be paid to the old generation, then the program $(\mathcal{Y}, \underline{\tau}(j), 0)$ wins the elections." In other words, when $\underline{\tau}(j)$ is the promised tax rate a realization of the growth rate $n(j+1)$ will trigger the end of the SSS. Notice that, for some $n(j)$'s, such a tax rate may not exist in $[0, 1]$.

ii. Let $n(j)$ be a growth rate for which a $\underline{\tau}(j)$ exists and call it the *least growth rate*. Taking $n(j)$ as given one can use the relation established in proposition 7.2 and proceed backward to determine the sequence of tax rates that yield an equality in each period.

iii. Finally let the first young generation choose the initial value $\tau(0)$. This is accomplished by selecting the best among all the sequences of $\tau(j)$ which were computed in the previous step. Notice that each such sequence (and hence each initial $\tau(0)$) corresponds to a different least growth rate.

The equilibrium we construct has a stationary nature: to each possible population growth rate is associated a tax rate, which is positive as long as the population growth is higher than a critical level, and then drops to zero. At that point the system collapses. When the population growth rate reaches the level which is immediately next to the critical rate the generation of young voters still prefers to keep the system going, even if they know that with positive probability they will pay and then will not be paid back by the next generation. Since the rate n_t is falling over time, it is not immediately clear how the equilibrium tax rates behave over time. We have:

Proposition 7.3 *The sequence of tax rates described in the previous proposition is tax rates is decreasing, that is*

$$\tau(j+1) < \tau(j)$$

for all j.

The details of the proof can be found in the appendix.

8 Conclusions

We have shown that a PAYG Social Security System may be supported as the subgame perfect equilibrium of an infinite horizon game in which economic agents choose the contribution and benefit rates by majority voting in every period and competitive markets determine saving and consumption levels. Neither altruistic motivations nor the existence of a benevolent social planner are needed in our model to explain the existence of PAYG pension plans.

A majority voting equilibrium may lead to the establishment of social security transfers from the young to the old even in circumstances in which the competitive equilibrium without such transfer would otherwise be converging to a consumption efficient steady-state. We conclude from this fact that, when social security policies are determined through voting, one may expect them to be often inefficient.

An interesting property of our model is the following. If a society faces an uncertain but asymptotically decreasing growth rate of the labor force, the majority voting equilibrium will lead to the disappearance of the PAYG SSS within a finite number of periods. The actual elimination of the SSS will be voted in at the time in which a certain lower bound on the growth rate of the labor force is realized. Nevertheless, we also show that in the meanwhile, i.e. until the least growth rate is not realized, it is perfectly rational for the median voter to maintain a SSS alive. In these circumstances the majority voting system leads to a non-increasing sequence of taxes and benefit rates.

The model also suggests a number of interesting questions worth investigating. One would be interested in classifying the efficiency properties of the political equilibria and the set of "constitutional restrictions" (if any) that might guarantee the maximization of one or another type of social welfare function by means of the majority voting system.

One would also like to verify the extent to which the political equilibrium with PAYG social security is modified by the introduction of income heterogeneity within each generation. A first step in this direction can be found in Tabellini (1990) who uses a static, two-period version of an OLG economy and shows that social security can be supported in a majority voting equilibrium by a coalition of old people and the poorest among the young ones. On the other hand, he does not examine the dynamic implications of this extension of the redistributional features of a PAYG system to the intragenerational level and the impact this may have on the set of intertemporal equilibria. By providing further incentives to redistribute income, such a modification would certainly lead to an increase in the size of the SSS as well as in the number of circumstances in which it may be adopted. On the other hand, the reduction in saving this would cause may be large enough to bring about a collapse of the system more often than in the circumstances we have considered in this article.

9 Appendix

9.1 Proofs

Proof of proposition ?? The SSS is sustainable as long as the values of V given by (4.6) and (4.8) are larger than the value of v given in (4.7). The difference $V - v$ is non-negative if and only if:

$$\log[(1-\tau_t) + \frac{1+n}{a}\tau_{t+1}] \geq 0 \quad \text{if} \quad s = 0 \tag{9.1}$$

and

$$\log(1-\tau_t) + \delta \log \tau_{t+1} + C \geq 0 \quad \text{if} \quad s > 0 \tag{9.2}$$

$$C \equiv \delta \log \left(\frac{1+n}{a\delta}\right) + (1+\delta)\log(1+\delta) \tag{9.3}$$

Also observe from (4.5) that

$$s \geq 0 \text{ if and only if } a\delta(1-\tau_t) \geq (1+n)\tau_{t+1}. \tag{9.4}$$

Inequality (9.1) is equivalent to

$$\tau_{t+1} \geq \frac{a}{1+n}\tau_t \tag{9.5}$$

and inequality (9.2) is equivalent to

$$\tau_{t+1} \geq e^{-\frac{C}{\delta}}(1-\tau_t)^{-\frac{1}{\delta}} \equiv \phi(\tau_t). \tag{9.6}$$

The function ϕ is increasing and convex, and

$$\phi(0) = e^{-\frac{C}{\delta}} > 0, \quad \phi'(0) = \frac{e^{-\frac{C}{\delta}}}{\delta}, \quad \phi(1) = +\infty. \tag{9.7}$$

So a fixed sustainable tax rate τ_0 exists if and only if

$$\tau^\delta(1-\tau)e^C \geq 1 \tag{9.8}$$

for some τ. The maximum of the function $\tau^\delta(1-\tau)$ is achieved at $\frac{\delta}{1+\delta}$, and from the expression for C we can compute:

$$e^C = (1+\delta)^{1+\delta}\left(\frac{1+n}{a\delta}\right)^\delta.$$

Substituting into (9.8) we get that this inequality is satisfied if and only if

$$\frac{1+n}{a} > 1.$$

Now note that at the tax rate $\frac{\delta}{1+\delta}$ the condition (9.4) for the savings to be positive is satisfied as equality; so it is satisfied at the smaller value τ_0, and is satisfied as a strict inequality when τ_0 is strictly smaller. So savings are positive at τ_0 when it exists.

For the last claim. Consider the map giving the minimum next period tax rate that makes V larger than v. This is defined by the two inequalities (9.5) and (9.6). When $a > 1+n$ the graph of this function, in both cases, is strictly above the diagonal. This proves our claim. ■

Proof of proposition ?? The value of the steady state capital stock is given by the equation (4.1). The value of the tax rate at which the two values $\frac{(1-\alpha)(1-\tau)}{1+n}$ and $\alpha\delta$ are equal is

$$\tau^* \equiv \max\left\{\frac{1-\alpha(1+\delta(1+n))}{1-\alpha}, 0\right\}. \tag{9.9}$$

From which we conclude that

i. for $\tau \geq \tau^*$ the value of the steady state capital is $(\frac{(1-\tau)(1-\alpha)}{1+n})^{\frac{1}{1-\alpha}}$, and the saving is equal to $(1-\tau)(1-\alpha)k^\alpha$, which is in turn equal to $k(1+n)$, using the expression for k;

ii. for $\tau \leq \tau^*$ the value of the steady state capital is $(\alpha\delta)^{\frac{1}{1\alpha}}$, and the saving is equal to $(1+n)(\alpha\delta)^{\frac{1}{1-\alpha}}$.

Recall equation (4.2), which here gives:

$$V(k(\tau), \tau, \tau) = (1-\tau)(1-\alpha)k^\alpha - s + \delta(1+n)^{1-\alpha}s^\alpha(\alpha + (1-\alpha)\tau).$$

Using this expression, in the first region ($\tau \geq \tau^*$), the value of the lifetime utility of each generation is found to be linear in the term $\tau[\delta(1+n)-1]$.

In the second region ($\tau \leq \tau^*$), the value is equal to $\delta[(1-\tau)w(k(\tau))\pi(k(\tau)) + w(k(\tau))(1+n)\tau]$. If we use the explicit form of k in this region, the value is (up to a positive multiplicative constant)

$$V = k^\alpha[\alpha + \tau(1-\alpha)].$$

Differentiating and using the fact that $\frac{dk(\tau)}{d\tau} = -\frac{k(\tau)^\alpha}{1+n}$ gives that this derivative is zero at $\tau = \frac{1-2\alpha}{1-\alpha}$. Our claim then follows. ■

Proof of proposition ?? A necessary condition for a sequence of tax rates τ_t to be the outcome of a political equilibrium is that the value of keeping the system is larger in each period, and every realization of the (n_t) process, than the value of dropping it. Recall that the second and third components of $\sigma_t(h_t)$, denoted τ^1 and τ^2, are the tax rates that the generation of young voters is setting for future periods in the event that the population growth rate does not change, or respectively does change. Since the equilibrium saving function equals the solution of the maximization problem of the representative young consumer, this inequality is equivalent to:

$$\max_{s\geq 0} \log[b(1-\tau_t)-s]+\delta(1-p)\log[as+b(1+n(j))\tau^1_{t+1}]+$$
$$+\delta p\log[as+b(1+n(j+1))\tau^2_{t+1}]\geq v;$$

where $n_t = n(j)$ and v, the value of dropping the SSS, is a constant defined in proposition 7.2, in the text.

To prove our claim, it is enough to prove that there is no solution to this infinite system of inequalities. Since the left hand side, keeping everything else fixed, is increasing in both $n(j)$ and $n(j+1)$, and the latter reaches almost surely in finite time any value lower than n_0 and higher than 1, it is enough to show that there is no solution to the same system when the population growth rate at t has reached a value n such that $a > 1+n$. So it is enough to prove that the system

$$\max_{s\geq 0} \log[b(1-\tau_t)-s]+\delta(1-p)\log[as+b(1+n_{t+1})\tau^1_{t+1}]+\delta p\log[as+b(1+n_{t+1})\tau^2_{t+1}]\geq v$$

does not have a solution extending in the infinite future. The latter defines implicitly a stochastic difference inclusion in the following way. Let first $\Phi(\tau_t)$ be the set of $(\tau^1,\tau^2)\in[0,1]^2$ such that, given τ_t, the inequality is satisfied.

The stochastic difference inclusion is defined now as: $\tau_{t+1}\in\Psi(\tau_t)$, where

$$\Psi(\tau_t)=\{\tau^1:(\tau^1,\tau^2)\in\Phi(\tau_t),\ \text{for some } \tau^2\},\ \text{if } n_{t+1}=n;$$

and

$$\Psi(\tau_t)=\{\tau^2:(\tau^1,\tau^2)\in\Phi(\tau_t),\ \text{for some } \tau^1\},\ \text{if } n_{t+1}<n.$$

We prove that for any path τ_t which is not identically zero the correspondence $\Phi(\tau_t)$ is empty valued in finite time, almost surely.

Let us list the properties of Φ that will be used in the sequel: 1. The set $\Phi(\tau)$ is convex; this follows from the fact that the function $\tau\mapsto\log(as+b(1+n)\tau)$ is concave. Also, it is easy to show by implicit differentiation that at the point of intersection between the diagonal in $[0,1]^2$ and the boundary of this set, the supporting line to the set has normal vector proportional to $(1-p,p)$. 2. Let ϕ be the function defined as

$$\phi(\tau)=\frac{a}{1+n}\tau, \qquad \text{when } \tau<\frac{\delta}{1+\delta}$$

$$\phi(\tau)=\exp\Big(\delta^{-1}(-C-\log(1-\tau))\Big), \qquad \text{when } \tau\geq\frac{\delta}{1+\delta}$$

where $C = -\delta \log(a\delta) + \delta \log(1+n) + (1+\delta)\log(1+\delta)$ The graph of ϕ lies entirely above the diagonal. Also, from the convexity of the image of Φ:

$$\Phi(\tau) \subseteq \Big\{\{(\tau^1, \tau^2) : (1-p)\tau^1 + p\tau^2 \geq 0\} + (\phi(\tau), \phi(\tau))\Big\}.$$

Addition of sets is defined as usual, element by element. Now from the fact that $\phi(\tau) > 1$ for τ large enough (precisely, for $\tau > 1 - e^C$), our claim follows. ■

Proof of proposition ?? Define

$$\Psi_1(\tau, n) \;\equiv\; \max_s \Big\{\log[(1-\tau)b - s] + \delta(1-p)\log[as + (1-n)\tau b] + \delta p \log as\Big\}$$

This is the lifetime utility achievable by a young generation starting with $n_t = n$ and $\tau_t = \tau$ and expecting to receive τ if $n_{t+1} = n$ and 0 if $n_{t+1} < n$. Notice that the equilibrium saving level $s^*(\tau)$ is positive if $(1-\tau)b > 0$ and that $s^*(0) = \frac{\delta b}{1+\delta}$. Of course $\Psi_1(0, n) = v$ for any n; but also the function Ψ_1 satisfies $\lim_{\tau \to 1} \Psi_1(1, n) = -\infty$ and

$$\frac{\partial \Psi_1}{\partial \tau}(0, n) = \frac{b\delta}{s^*(0)}\left[\frac{(1-p)(1+n)}{a} - 1\right] > 0$$

if and only if $\frac{a}{1+n} < 1-p$. Our assumptions imply that this is true for a non-empty but finite set of population growth rates in the sequence $n(j)$. Now if we take any such $n(j)$ to be the value of n in the function Ψ_1, we conclude that a $\underline{\tau}(j) > 0$ exists for any j such that $\Psi_1(\underline{\tau}(j), n(j)) = v$.
We move now to the backward induction construction of the tax rates. Take $n(j)$ and $\underline{\tau}(j)$ as defined in the previous step. Recall that this implies that when $n(j+1)$ is realized the equilibrium tax is zero and the system is terminated. For $i = 1, 2, \ldots$ assume that the value of $\tau(j-i)$ which solves

$$\max_s \;\Big\{\log[(1-\tau(j-i))b - s] + \delta(1-p)\log[as + (1+n(j-i))\tau(j-i)b] +$$
$$+\delta p \log[as + (1+n(j-i+1))\underline{\tau}(j-i+1)b]\Big\} = v$$

has been found. We look for the next value $\tau(j-i-1)$ to be associated to $n(j-i-1)$ by solving

$$\Psi_2(\tau, n(j-i-1)) - v = 0$$

where

$$\Psi_2(\tau, n) \;\equiv\; \max_s \;\Big\{\log[(1-\tau)b - s] + \delta(1-p)\log[as + (1+n)\tau b] +$$
$$+\delta p \log[as + (1+n(j-i))\underline{\tau}(j-i)b]\Big\} = v$$

Note that

$$\Psi_2(1, n) = -\infty$$

and

$$\Psi_2(0, n) > \max_s \;\log[b - s] + \delta \log as = v$$

hence a solution $\tau(j-i-1)$ always exists. Repeating this procedure for all $i = 1, \ldots, j-1$ and all $n(j)$ generates a the required sequence of non-zero tax rates. ∎

Proof of proposition ?? As usual given the equality between the value of keeping the social security system and the value of dropping it, a sequence of equilibrium strategies is easy to define, and is given in the main text.

We turn to the real issue, of proving that the equilibrium sequence of tax rates is decreasing over time; in fact, we prove the statement for any sequence $(\tau(j)$ satisfying the equality stated in proposition **??**. First we fix some notation: j_0 is the index for the least growth rate to which a positive tax rate is associated; so $\tau(j_0) > 0$, and $\tau(j) = 0$ for any $j \geq j_0 + 1$. We also define the function $\Psi(s; n(j), \tau(j), n(j+1), \tau(j+1))$ as:

$$\Psi(s; n(j), \tau(j), n(j+1), \tau(j+1)) = \log[(1-\tau(j))b - s] +$$

$$+\delta(1-p)\log[as + (1+n(j))\tau(j)b] + \delta p \log[as + (1+n(j+1))\tau(j+1)b].$$

For future use, we now note an obvious but important property of Ψ. The function $(s, \tau) \mapsto \Psi(s; n(j), \tau, n(j+1), \tau(j+1))$ is concave, since it is the composition of concave and linear functions. Therefore the function: $\tau \mapsto \max_{s \geq 0} \Psi(s; n(j), \tau, n(j+1), \tau(j+1))$ is also concave.

We now proceed with the proof of our main claim. The proof is by induction on the index j. We begin with the inequality

$$\tau(j_0) \leq \tau(j_0 - 1).$$

Note first that for any s and τ the following inequality is immediate from the definition of Ψ: (recall that $\tau(j_0 + 1) = 0$):

$$\Psi(s; n(j_0), \tau, n(j_0+1), \tau(j_0+1)) \leq \Psi(s; n(j_0-1), \tau, n(j_0), \tau(j_0)).$$

Hence the same inequality is preserved by taking $\max_{s \geq 0}$ on both sides: this operation gives us two functions of τ, Ψ_{j_0} and Ψ_{j_0-1}, say, with the first function less or equal to the second, pointwise over $[0,1]$. We have already seen that these two functions are equal to v for at least one value of τ (in fact, we have already substituted one of these values, $\tau(j_0)$, in the corresponding Ψ). From the fact that they are concave we now know that this value is unique; and from the inequality between the two functions we know that the value for the function Ψ_{j_0}, $\tau(j_0)$ is less or equal to the value of the function $\Psi(j_0 - 1)$ at $\tau(j_0 - 1)$.

Now for the other tax rates. By the induction hypothesis we have the inequality $\tau(j+1) \leq \tau(j)$, and we now claim that $\tau(j) \leq \tau(j-1)$. The induction hypothesis, together with the inequality on growth rates of population, gives the inequality:

$$\Psi(s; n(j), \tau, n(j+1), \tau(j+1)) \leq \Psi(s; n(j-1), \tau, n(j), \tau(j))$$

for any τ. The argument now is identical to the one given in the first step, and we conclude our proof. ∎

9.2 Definitions

Here we provide the relevant definitions for the model with stochastic population growth rates. They follow closely the lines of the definitions for the deterministic model. A *tax process* is a measurable function from the history of the economy to $[0, 1]$.

Definition 9.1 *For a given pair of processes of population growth rates and tax rates, a* **competitive equilibrium** *is a process* $(w_t, \pi_t, c_t^t, c_{t+1}^t)$ *such that, almost surely:*

i. the equilibrium conditions for the firms in section 2 above are satisfied;

ii. the saving process (s_t) *maximizes the expected utility of the representative young agent, conditional on the history;*

iii. markets clear.

On the basis of this, we say that:

Definition 9.2 *A* **generation player game** *is the extensive form game where*

i. players are indexed by $t \in (1, 2, \ldots)$*;*

ii. the action set of each player is $\{\mathcal{Y}, \mathcal{N}\} \times [0, 1]^2$*;*

iii. for every history of the population growth rates and actions, the payoff to the generation player t *is the expected utility at the corresponding competitive equilibrium conditional on that history.*

The element (Y, τ_1, τ_2) of the action set is to be interpreted as follows: the generation of young players accepts to pay the tax rate set in the previous period, and sets a transfer rate (to themselves) of τ_1 in the event that the population growth rate stays constant, and τ_2 otherwise.

We also remark that histories are now a list of past actions of the generation players, and of the outcome of the population growth rate in the past; to be precise by a history at time t we mean an element of the form $(n_1, a_1, \ldots, a_{t-1}, n_t)$. Finally we say that:

Definition 9.3 *For a given stochastic process of the population,* (n_t)*, a* **political equilibrium** *is a process* $(\tau_t, w_t, \pi_t, c_t^t, c_{t+1}^t)$ *such that:*

i. the process $(w_t, \pi_t, c_t^t, c_{t+1}^t)$ *is a competitive equilibrium for the given population and tax processes;*

ii. there exists a sequence (σ_t) *of strategies of the generation player game which is subgame perfect equilibrium and gives as outcome the process* (τ_t)*.*

References

[1] Azariadis, C. and Galasso, V. (1997), “Fiscal Constitutions and the Determinant of Intergenerational Transfers”, Universidad Carlos III de Madrid, working paper 97-71.

[2] Becker, G. and Murphy, K. (1988), “The Family and the State”, *Journal of Law and Economics*, **XXXI**, 1-18.

[3] Boadway, R.W. and Wildasin, D.E. (1989), “A Median Voter Model of Social Security”, *International Economic Review*, **30**, 307-327.

[4] Boldrin, M. and Rustichini, A. (1995), “Equilibria with Social Security”, mimeo, Universidad Carlos III and C.O.R.E., Louvain.

[5] Browning, E. (1973), “Social Insurance and Intergenerational Transfers”, *Journal of Law and Economics*, **16**, 215-237.

[6] Browning, E. (1975), “Why the Social Insurance Budget is Too Large in a Democracy”, *Economic Inquiry*, **XIII**, 373-388.

[7] Cooley, T.F. and Soares, J. (1998), “A Positive Theory of Social Security Based on Reputation”, *Journal of Political Economy*, **107**, .

[8] Diamond, P. (1965), “National Debt in a Neoclassical Growth Model”, *American Economic Review*, **55**, 1126-1150.

[9] Diamond, P. (1977), “A Framework for Social Security Analysis”, *Journal of Public Economics*, , 275-298.

[10] Esteban, J.M. and Sákovics, J. (1993), “Intertemporal Transfer Institutions”, *Journal of Economic Theory*, **61**, 189-205.

[11] Feldstein, M. (ed., 1987), *Taxes and Capital Formation*, Chicago, IL.: The University of Chicago Press.

[12] Galasso, V. (1998), “ The US Social Security System: What Does Political Sustainability Imply?” forthcoming, *Review of Economic Dynamics.*

[13] Hammond, P. (1975), “Charity: Altruism or Cooperative Egoism.”, in E.S. Phelps (ed.) *Altruism, Morality and Economic Theory*, New York, NY: Russell Sage Foundation.

[14] Kandori, M. (1992), “Repeated Games Played by Overlapping Generations of Players”, *Review of Economic Studies*, **59**, 81-92.

[15] Kotlikoff, L.J. (1987), “Justifying Public Provision of Social Security”, *Journal of Policy Analysis and Management*, , 647-689.

[16] Lazear, E. (1979), "Why is there Mandatory Retirement?", *Journal of Political Economy*, **87**, 275-296.

[17] Merton, R.C. (1983), "On the Role of Social Security as a Means for Efficient Risk Sharing in an Economy where Human Capital is not Tradeable", in Z. Bodie and J. Shoven (eds.) *Financial Aspects of the United States Pension System*, Chicago, IL.: The University of Chicago Press for the N.B.E.R.

[18] Sala-i-Martin, X. (1992), "Transfers", N.B.E.R. w. p. No. 4186, October.

[19] Salant, S. (1991), "A Repeated Game with Finitely Lived Overlapping Generations of Players", *Games and Economic Behavior*, **3**, 244-259.

[20] Samuelson, P. (1975), "Optimum Social Security in a Life-cycle Growth Model", *International Economic Review*, **16**, 539-544.

[21] Sjoblom, K. [1985], "Voting for Social Security", *Public Choice*, **45**, 225-240.

[22] Tabellini, G. (1990), "A Positive Theory of Social Security", N.B.E.R. w. p. No. 3272, September.

[23] Verbon, H.A.A. (1987), "The Rise and Evolution of Public Pension Systems", *Public Choice*, **52**, 75-100.