Persistent Oscillations and Chaos in Dynamic Economic Models: Notes for a Survey

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1 Introduction

It is probably not unfair to say that Nonlinear Dynamics (NLD) has not had a major impact on the development of modern economic theory. In fact, one may even be tempted to add that, until very recently, it was either an unfamiliar tool for the mathematical economist or one whose implications were often disregarded as irrelevant to the purposes of the research. Dynamical systems theory appeared for a while in the background of the studies on the stability of the tâtonnement process (see Hahn, 1982) and on optimal growth and turnpike (see McKenzie, 1986), but never really got on the stage.

During the 1950's and 1960's the only well-developed effort to use nonlinear techniques in the study of dynamic economic processes is associated with Richard Goodwin (see Goodwin, 1982, for a collection of the relevant essays). He put forward the idea of illustrating persistent, deterministic oscillations within a multiplier-accelerator setup by means of a limit cycle for a nonlinear, two-dimensional flow. His research effort toward an endogenous explanation of economic fluctuations motivated a few others' contributions within the Keynesian and Cambridge (UK) tradition, but was never able to take off and influence the whole of the profession. In fact, the late 1960's and 1970's witnessed an almost complete unanimity on the use of linear-stochastic models in order to understand business cycles. The causes of this historical process are complex and they will not concern us in this place. Similarly, I will not try to list the motivations for the sudden revival of interest in NLD which is characteristic of the last few years, but simply try to provide a (cursory) description of the problems that have been considered and of the results achieved. At the end, I will also take a timid look at the “mare magnum” of issues that are still open to investigation.

Let me make clear, at the outset, that the revival of interest in NLD to which I refer is not widespread among economists and that, as a matter of fact, the great majority of applied and theoretical researchers still look at it with doubtful eyes and believe it will not help much in understanding what is going on in the real economic world. They may, indeed, be right.

In spite of this, a group of scholars have taken the opposite perspective and have started to look at economic fluctuations under the hypothesis that a relevant portion of them can be explained as a deterministic phenomenon, endogenously created by the interaction of market forces, technologies and preferences. In particular, it is conjectured that deterministic periodic cycles affected by small stochastic forces and/or “noisy” chaotic paths generated by dynamical systems of relatively small dimensionality, can account for a relevant portion of the observed fluctuations of most of the important macroeconomic variables.

Such a research program naturally involves two lines of inquiry: (1) finding theoretical models that predict cycles and chaos as logical outcomes of “reasonable” economic hypotheses and (b) testing the available data in order to find evidence of nonlinearities in the underlying dynamic processes. A third, and most important, step should indeed be added: comparing the data and the qualitative predictions of the theoretical models in order to understand if they are at least compatible or if instead the former reject the latter. Such a task remains, for the time being, far from being undertaken. The available models are too abstract to yield any seriously testable implication and the data-screening techniques we know of seem still too weak to place any confidence in their results. In any case, it is clear that the approach we are discussing will win or lose its bet exactly on this point; any effort along those lines is, therefore, worthwhile.

This survey (fortunately!) is limited to point (a); issues and results that pertain to (b) are illustrated in Brock (1987; see also Ruelle, 1987). I have tried to the best of my ability to make this an economic survey. Given the intrinsically technical nature of the problems and the sophisticated mathematics involved, this choice entails a major consequence: I have dispensed completely with mathematical definitions, lemmas and theorems, *taking as granted* that the reader is familiar enough with the terminology to have at least an intuition of what is going on from a mathematical point of view. I have assumed the reader knows the mathematics of chaos and wants to learn a bit about the economics of chaos. For the non-technical reader, I can only recommend a few standard references: Collet-Eckman (1980), Devaney (1986), Guckenheimer-Holmes (1983), Iooss (1979) and Lasota-Mackey (1985).

The note follows these steps: in the second section I discuss the earlier examples of competitive models having an oscillatory or chaotic dynamics. Those of them which had a macroeconomic structure were not derived from explicit maximizing behavior on the part of the agents. As it used to be believed that intertemporal maximization would eradicate instability, we consider in the next two sections models with a complicated dynamics which are grounded on rational, maximizing procedures. Section 3 takes care of the so-called overlapping generations model (OLG) and Section 4 examines the models where agents live forever. Section 5 gives some concluding comments. The bibliography at the end is meant to include all the works on nonlinear dynamics in economics which have a theoretical nature, have been published during the "revival" and are known to me.

2 Non-Optimizing Models of Economic Dynamics

2.1 Keynes-Kaldor Models

At the outset, there was the "Keynesian" model, in one of its many possible dynamic versions. Torre (1977) for example considered:

$$\begin{aligned} \dot{Y} &= \alpha\{I(Y,R) - S(Y,R)\} \\ \dot{R} &= \beta\{L(Y,R) - L^S\} \end{aligned} \tag{2.1}$$

where Y is real income, R is the rate of interest; I and S are the investment and saving function with $\partial I/\partial Y > 0$, $\partial I/\partial R < 0$, $\partial S/\partial Y > 0$, and $\partial S/\partial R > 0$; L is the demand for money with $\partial L/\partial Y > 0$ and $\partial L/\partial R < 0$; and L^S is the fixed money supply. Finally, α and β are two positive parameters. That such a model could exhibit a limit cycle was a kind of folk theorem to which Torre provides a formal proof by showing that an Hopf bifurcation occurs when the steady state loses stability as the bifurcation parameters α and β increase. This came as no surprise: the model was formally analogous to the standard representation of Kaldor's business-cycle model, which was known to have limit cycles as a solution since the work of Chang and Smith (1971). This is in fact written as:

$$\begin{aligned}\dot{Y} &= \alpha\{I(Y,R) - S(Y,K)\}\\ \dot{K} &= I(Y,K) - \beta K\end{aligned} \tag{2.2}$$

where K is the aggregate capital stock; β is the depreciation factor; and the functions satisfy $\partial I/\partial Y > 0$, $\partial I/\partial K < 0$, $\partial S/\partial Y > 0$, and $\partial S/\partial K \neq 0$.

Dana and Malgrange (1984) contains an interesting analysis of the model of Eq. (2.2). After proving the existence of a limit cycle for a continuous-time parametric version of the model (with parameter values specified to satisfy French quarterly data for 1960–74), the authors addressed, using both simulation techniques and analytical instruments, a discrete-time version of the same model. They showed that, by using α as a bifurcation parameter, different regimes may be obtained that go from attracting steady state to an apparently chaotic state that they label “intermittent chaos.” Even if they were unable to prove the existence of a strange attractor for their model, the two authors provided a considerable amount of evidence to this end.

Both the “Keynesian” and the “Kaldorian” models are susceptible to various criticisms. A couple of them may be handled by means of the Hopf bifurcation. The first relates to the fact that you need to make strong global assumptions on the shape of I or S to obtain a cycle in Eq. (2.2) (or Eq. (2.1)) by means of the Poincaré-Bendixson theory. In fact, Chang and Smith (1971) had to assume either I or S to have an S-shape with respect to Y, for given values of K, a strong assumption with little support. By using Hopf, you may disregard this hypothesis; all you need are local information on the degree of sensitivity of I to changes in Y and K (or R) around the steady state.

A second criticism is concerned with the fact that in both Eq. (2.1) and Eq. (2.2), an important variable that does change along the business cycle is considered fixed by the model: this is K in Eq. (2.1) and R in Eq. (2.2). This means that you would like to consider a more complete model like:

$$\begin{aligned} \dot{Y} &= \alpha\big\{I(Y,R,K) - S(Y,R,K)\big\} \\ \dot{R} &= \gamma\big\{L(Y,R) - L^S\big\} \\ \dot{K} &= I(Y,R,K) - \beta K \end{aligned} \tag{2.3}$$

where the signs of the first partial derivatives are as before. The Hopf bifurcation is especially useful here as the standard Poincaré-Bendixson technique is of no help in high dimensions. The model of Eq. (2.3) has been studied by Boldrin (1983) and Cugno-Montrucchio (1983): the existence of limit cycles is proved for large sets of values of the parameters. In fact, Cugno and Montrucchio add also a fourth variable (price expectations) and still obtain oscillatory solutions by means of the Hopf theorem. Almost no one seems to have looked for the emergence of more complicated patterns of behavior in Eq. (2.3), even if it seems obvious that a bifurcation cascade may easily be originated and, therefore, the Newhouse-Ruelle-Takens (1978) theory applied to argue the existence of strange attractor: the thing seems, indeed, to be computationally demanding and with a very low return for our economic understanding. Lorenz (1985) has tried something along these lines for a six-dimensional version of Eq. (2.2) (three different goods and three different capital stocks), but he does not seem to get any economic insight. Along similar lines, a related work is that of Medio (1984) which considers an n-dimensional multiplier-accelerator model and proves the existence of cycles by using both the ideas of synergetics and the Hopf bifurcation theorem.

Still within a Keynesian framework, there is another attempt to explain the emergency of complicated dynamics that is worth mentioning. This is Day-Shafer (1985). A modified version of the textbook IS-LM approach provides a consumption income function and an investment-income function:

$$C = G(Y, M) = C\big[L(Y, M), Y\big]$$
$$I = \alpha H(Y, M) = \alpha I\big[L(Y, M), Y\big] \tag{2.4}$$

where M denotes the amount of money, and and $L(Y, M)$ is the LM function which expresses the value of the interest rate R that guarantees a temporary money-market equilibrium for given Y and M. The other notation is as before. A dynamic process is obtained by using what is called a Robertsonian lag (current C and I depend on past income) that yields a nonlinear version of Samuelson's multiplier-accelerator process:

$$Y_{t+1} = \theta(Y_t; \alpha, M, A) = G(Y_t; M) + \alpha H(Y_t; M) + A. \tag{2.5}$$

A is a positive constant representing exogenous expenditure. The money supply here is taken as a parameter; the dynamic is therefore described by the one-dimensional map of Eq. (2.5). As the authors wanted to use the existing theory of unimodal maps, they needed to make θ unimodal. Consumption is monotonically increasing in income (with a slope bounded away from zero and bounded above by 1); therefore, the burden of nonmonotonicity is on the investment function. Their intuition goes as follows: investment demand increases with income for a low level of income because of the accelerator mechanism and low interest rates. But as income increases, the fixed amount of money supplied requires an ever-rising interest rate to clear the money market. If investment is elastic enough at high interest rates, then it must eventually fall. Also, as α grows, the hump in θ increases and it is clear that by treating it as a bifurcation parameter, the classical route to chaos can be obtained for fairly simple specifications of G and H. In fact, the authors accomplished this by using piecewise continuous maps. Even more, they showed that the accelerator mechanism is not strictly necessary to the argument: even if I depends only on R, the negative effect of Y for high Y's can be brought in by the demand for money functions that determines the market interest rate. One may notice that models of this type leave plenty of room for stabilizing (or destabilizing) monetary policies. Leaving expectations considerations aside, it is clear that a procyclical monetary policy could inject the amount of money necessary to prevent R from increasing as Y increases and, therefore eliminate the unimodal shape of θ or, at least, reduce its steepness. A similar, but not identical point is made in Day (1984). On the nonlinear accelerator, one may also want to look at Rustichini (1983), even if I must admit that the sense of the latter escapes my understanding.

For completeness, we may also recall the work of Simonovits (1982). He adopts the framework introduced by Benassy and Malinvaud of distinguishing between "classical" and "Keynesian" equilibria in the wage-price space and tries to describe a dynamical system in those variables. The author claims the existence of cycles and chaos but nothing precise is actually proven.

2.2 Class Struggle and Chaos

A role was played in the earlier debate by a simple and elegant model that Richard Goodwin elaborated to formalize Marxian or conflictual views of economic growth and income distribution. Even if the huge amount of research done around this model has had little or no spillover on any other area of economic dynamics, the basic idea is very interesting by its own sake and worth mentioning. The original setup (see Goodwin, 1967) is like this: q is output, k is capital stock, w is the wage rate, $a = a_0 \exp(\alpha t)$ is labor productivity trend, $\sigma = k/g$ is a fixed ratio, $u = w/a$ is the labor share in national income, and $(1 - w/a)a = k$ are profits that are completely invested. If the work force grows like $n = n_0 \exp(\beta t)$ and $\ell = q/a$ represents the employment level, time differentiation will give:

$$\frac{\dot{v}}{v} = \left(\frac{\ell - u}{\sigma}\right) - (\alpha + \beta) \tag{2.6}$$

where $v = \ell/n$ (employment ratio). Assume wages vary according to $\dot{w}/w = f(v) = -\gamma + \rho v$ (the bargaining rule), then time differentiation will also give:

$$\frac{\dot{u}}{u} = -(\alpha + \gamma) = \rho v. \tag{2.7}$$

Eqs. (2.6) and (2.7) together give a bi-dimensional dynamical system well known to mathematical biologists, i.e., Lotka-Voltera “prey-predator” model.

The solution to it is given by a continuum of closed curves around the unique stationary state. Different initial conditions will place the system on different oscillatory paths along which profits/wages and unemployment oscillate perpetually. The economic interpretation is obvious, given the premises.

Pohjola (1981) modified the model in order to use the one-dimensional setup for maps. Beside the translation of the basic relations into a discrete-time version, he modified the wage-bargaining rule assuming that the level employment of determines the wage level and not its rate of change. In our notation, this reads:

$$u_t = -\gamma + \rho v_t \tag{2.8}$$

and the state of the model is fully described by the single variable v_t, the employment ratio. The dynamical system is now:

$$v_{t+1} = v_t \left[1 + A \left\{1 - \frac{v_t}{v^*}\right\}\right] \tag{2.9}$$

where

$$A = \frac{1 - \sigma(\beta + \alpha + \alpha\beta) + \gamma}{\sigma(1 + \beta + \alpha + \alpha\beta)}$$

and

$$v^* = \frac{1 - \sigma(\beta + \alpha + \alpha\beta) + \gamma}{\rho}.$$

A simple change of variable will transform Eq. (2.9) in the quadratic map $x_t = \alpha x_{t-1}(1 - x_{t-1})$, which has well-known chaotic properties.

2.3 Descriptive Growth Models

Most of the research efforts illustrated so far went unnoticed outside the boundaries of a restricted number of "aficiniados." A somewhat wider attention was attracted by Day (1982). His starting point is a capital-accumulation equation of the form:

$$k_{t+1} = \frac{s(k_t) \cdot f(k_t)}{1+\lambda} \tag{2.10}$$

where s is the saving function, f the production function and $\lambda > 0$ is the exogenous population's growth rate. Thi is a discrete-time version of the famous Solow's growth model. The latter had used a continuous-time specification to show that under neoclassical assumptions, any capital accumulation path will converge to a steady-state position. Day, on the reverse, exploits the well-known instability of unimodal maps to provide examples of chaotic behavior within that very same framework. As the model is not an optimizing one, i.e., the aggregate saving function is not explicitly derived from considerations of intertemporal efficiency, the author is free to pick "reasonable" shapes for $s(k_t)$ (and $f(k_t)$ obviously) in order to prove his claim. He begins with a constant saving ratio σ and a Cobb-Douglas form for f; Eq. (2.10) then becomes

$$k_{t+1} = \frac{\sigma B k_t^{\beta}}{1+\lambda} \tag{2.11}$$

which is monotonic and therefore stable. By introducing a "pollution effect" in the production function, one obtains

$$k_{t+1} = \frac{\sigma B k_t^{\beta}(m - k_t)^{\gamma}}{1+\lambda} \tag{2.12}$$

which is unimodal and has period-three for certain ranges of parameter values. Returning to the Cobb Douglas form and allowing instead for a variable saving rate, $s(k) = a(1 - b/r)k/y$, he obtains

$$k_{t+1} = \left[\frac{\phi}{1+\lambda}\right] k_t \left[1 - \frac{b}{\beta B} k_t^{1-\beta}\right] \tag{2.13}$$

using the fact that the rate of interest must be $r = \beta y/k$. This equation also displays topological chaos for feasible parameter values.

It is obviously very easy to question the empirical validity of this exercise, but this will not eliminate the simple fact that Day showed: small perturbations of well-established models could yield dynamic predictions that go in the direction of chaos. As chaotic dynamics can be displayed by such a simple, basic model, there are no reasons to sustain the claim that it is theoretically irrelevant.

Growth models appear to be particularly well suited to provide examples of economic chaos. Day himself has worked out the chaotic properties of certain formalizations of classical (Malthus) theories of economic growth (see Day, 1983, and Baduri and Harris, 1987, for the Ricardian system). Stutzer (1980), in turn, had already considered the growth model of Haavelmo and translated it in a discrete-time version that, again, was homeomorphic to the quadratic map. The original model he starts from is a simple one-dimensional differential equation with a globally stable state. Once translated into discrete time, the very same system becomes chaotic. This fact occurs often in many of the first works on chaos in economics and casts some doubt on the relevance of the findings. Indeed, one would like the qualitative results of a model to be invariant with respect to changes from continuous to discrete time. If this does not happen, then we are entitled to question the appropriateness of the chosen formalism as well as the relevance of the result. We may claim, in fact, that the change in time units introduces hidden assumptions into the model (lags, for example) and that these should be properly clarified. The issue has been scarcely considered by the theorists working in the field and it does not seem easily solvable. In any case, it is true that we have to look with some suspicion at those results that depend almost entirely on the discreteness of the time representation and that cannot be replicated in continuous time. Because one-and two-dimensional autonomous vector fields cannot produce chaos, this would imply that a "natural translation" of the low-dimensional map into a higher dimensional flow should be possible and should also preserve the qualitative predictions of those models we want to consider seriously as an explanation for dynamic economic complexities.

A second, more direct, criticism can be used against most of the results presented so far: they are derived at an aggregated, macroeconomic level, assuming some kind of competitive behavior on the part of the market participants, but without ever spelling out the kind of objectives these agents are pursuing and the set of constraints within which they are bound to operate. In short, all of the previous macromodels lack a microeconomic foundation in terms of explicit intertemporal optimizing behavior. Clearly, one can always reject such a criticism as irrelevant by claiming that actual economic agents do not, in fact, maximize consistently over time and follow instead adaptive paths and rules-of-thumb in making their decisions. Such a position has been taken, among others, by Richard Day on more than one occasion (see Day, 1986, for all of them), and it solves the problem by removing it to a different level of analysis.

As a matter of fact, the "rationality assumption" is instead taken very seriously, for one reason or another, by the majority of economists (myself included) and is therefore worth some investigation. The criticism will turn out to be wrong but it possesses, indeed, a strong intuitive basis. Economic agents (either consumer-workers or firms) are typically assumed to maximize concave objective functions over convex feasible sets. As the concavity refers to variables indexed by time, this would suggest that cycles (and *a fortiori*, chaos) should not be optimal. Averaging over cycles will exploit the concavity of the function and therefore increase the value achieved. One may conjecture from this reasoning, and it will become clearer from the discussion of Section 4 that the critical assumption behind the argument is not rationality per se, but rationality and concavity together.

In any case, the rationality-based criticism is widely spread among economists even if very seldom has it been expressed in written form. In one case (Dechert, 1984), it appeared to be especially strong and well grounded, particularly because it was independent from the concavity assumption. Dechert's argument goes as follows: pick, for example, Day's version of the Solow growth model and ask if the saving function he is using in his example could be determined, everything else being equal, as the solution to a representative-agent, infinite-horizon maximization problem. The answer is negative. More formally, let $y_t = f(k_t)$ be total output at time t, as a function of the existing stock of capital. The consumer-producer chooses how to split it between consumption and future capital in order to maximize $\sum_{t=0}^{\infty} u(c_t)\delta^t$, where u is a concave utility function, δ is a time-discount factor, $\delta \in (0,1)$, and k_0 is given as an initial condition. It turns out that, even if the production function is not concave, the optimal program $(k_0, k_1, k_2, \dots)$ can be experssed by a policy function $k_{t+1} = \tau(k_t)$ which is monotonic. The dynamical system induced in this way cannot therefore produce cycles or chaos. The economic prediction is that such a society will asymptotically converge to some stationary position. The latter is unique when f is concave. From this we have to conclude that the chaotic examples derived from a one-sector growth model would not pass the rationality critique. Such a critique turns out to be rather special itself, as it holds true only for the special version of the one-sector growth model considered above. This will be illustrated in the next two sections.

2.4 Miscellanea

A variety of different economic models have been considered in recent years in order to show that they could admit, under reasonable hypotheses, chaotic outcomes. Albin (1987) considers a disaggregated model of firms' interaction that gives rise, at the aggregated level, to the behavior considered in Day (1982); many interesting simulations are provided. Baumol and Benhabib (1987) contains an introductory survey to chaos in economics, while Baumol and Wolff (1983) prove chaos in a simple model of research and development. Benhabib and Day (1981) is an extremely interesting paper in which an axiomatic approach is taken to dynamic consumer behavior: a set of conditions is imposed on preferences and the way they depend on experience in order to produce chaotic consumption paths in an environment where income and prices are fixed. Deneckere and Judd (1986) prove that the innovation dynamic may be chaotic under certain patent rules. Jensen and Urban (1982) show chaos for the old cobweb dynamics.

In a very nice paper, Rand (1978) first introduced these arguments in dynamic games by showing how a natural duopolistic interaction (of the Cournot-Nash type) could lead to chaos. In Dana-Montrucchio (1986) another class of infinite-horizon repeated games is considered: it is shown that chaos and almost everything else are possible Markov perfect equilibria.

Finally, let us recall the collective volume edited by Jean Michel Grandmont

(1987) which contains some of the most significant recent papers in this area; these were presented at a conference held in Paris in June 1985 and first published in the October 1986 issue of the *Journal of Economic Theory*. We will survey some of these works in the following sections, but a direct look at the whole volume may help the interested reader to put the material in a more proper perspective.

3 Overlapping Generations Models

To overcome the "lack of rationality" critique, we need to place our economic agents in an environment where they have to make a concrete intertemporal consumption-investment choice in order to achieve a well-specified objective in the face of given or expected prices and resource constraints. The simplest framework is provided by a model of overlapping generations with constant population and a representative agent per generation and an exogenously specified endowment stream of the consumption good. Let's indicate with a superscript y those variables pertaining to the youngs and with o those for olds, let $t = 0, 1, 2, \ldots$ indicate calendar time. Preferences are represented by a utility function $U(c_t^y, c_{t+1}^o)$, where c_t^y is consumption when young of an agent born at y, and c_{t+1}^o is the same agent's consumption when old. Finally, let (w^y, w^o) denote the time-invariant endowment pair and p_t the price of the homogeneous good at time t, so that $\rho_t = p_t/p_{t+1}$ is the interest factor at time t. The representative agent will maximize his lifetime utility

$U(c_t^y, c_{t+1}^o)$ under the budget constraint:

$$c_{t+1}^o = w^o + \rho_t[w^y - c_t^y]. \tag{3.1}$$

Standard concavity assumption will give two utility maximizing consumption demands: an intertemporal competitive equilibrium will then be a sequence of vectors (ρ_t, c_t^y, c_t^o) such that utility of each generation is maximized under Eq. (3.1) and the material balance constraint:

$$[w^y - c_t^y] + [w^o - c_t^o] = 0 \tag{3.2}$$

is also satisfied.

Clearly the youngs can either save or borrow and therefore they may carry claims or debts into the second period. Assume this is done by means of a universally accepted paper asset called money (or checking account). Following Gale (1973), let's call "classical" the case in which the young are impatient and borrow and "Samuelson" the opposite one. Which state will occur clearly depends both on the shape of the utility function U and the relative magnitudes of w^y and w^o. Note also that the no-exchange (and no-money) equilibrium is always a possible outcome and it is such that if it obtains in the first period it will be replicated forever as our economy is time-invariant in its fundamentals. Gale also showed that, under the natural dynamics we will introduce in a moment, such autarkic equilibrium is locally unstable in the classical cases and locally stable in the Samuelson economies.

Benhabib and Day (1982) were the first to consider these economies from the point of view of nonlinear analysis (even if Gale has already pointed out the possibility of cycles). They studied the dynamics of the classical case (in an earlier paper Benhabib and Day, (1980), they had used the overlapping generations model with capital and production to obtain chaos, but that result was critically dependent upon a rather questionable use of a future-utility discount factor varying positively with wealth). I will briefly summarize their analysis in the following pages.

Assume all solutions are interior. Using the first-order necessary and sufficient conditions for utility maximization together with the budget constraint, one obtains the equality:

$$\frac{U_1(c_t^y, c_{t+1}^o)}{U_2(c_t^y, c_{t+1}^o)} = \frac{w^o - c_{t+1}^o}{c_t^y - w^y} \tag{3.3}$$

where U_1 and U_2 are the partial derivatives of U. Under regularity assumptions, Eq. (3.3) can be solved uniquely for c_{t+1}^o, call this function

$$c_{t+1}^o = G(c_t^y; w^y, w^o). \tag{3.4}$$

Now use Eq. (3.4) to eliminate c_{t+1}^o from the left-hand side of Eq. (3.3) which, in equilibrium, must be equal to ρ_t. Let's call this newly obtained ratio the constrained marginal rate of substitution (CMRS), it will be a function of c_t^y only and of the parameters w^y, w^o. Denote it by $V(c_t^y; w^y, w^o)$. Finally, use the latter together with the material balance A of Eq. (3.2) to obtain a first-order difference equation in the youngs' consumption levels:

$$c_{t+1}^y = w^y + V(c_t^y; w^y, w^o)(c_t^y - w^y) \equiv f(c_t^y). \tag{3.5}$$

The problem is now that of providing conditions for $f(c_t^y)$ to be unimodal and with the degree of steepness sufficient to produce chaotic trajectories. The authors looked for chaos in the "topological" sense (i.e., existence of a period-three orbit), but in fact provided examples of utility functions and endowment pairs for which also the stronger form of chaos (i.e., existence of an invariant, absolutely continuous and ergodic measure) can be obtained.

Naturally such sufficient conditions depend on U, through the CMRS, and amount to saying that $V(c_t^y)$ can vary sufficiently over the interval $I = (w^y, w^y + w^o)$. These conditions are: there exists a $\hat{c} > w^y$ such that:

$$\alpha_1 = V(\hat{c}) > 1 \qquad \text{(resp. } < 1) \tag{3.6a}$$

$$a_2 = V\big(\alpha_1 \hat{c} + (1 - \alpha_1) w^y\big) > 1 \qquad \text{(resp. } < 1) \tag{3.6b}$$

$$0 < \alpha_3 = \alpha_1 \alpha_2 V\big(\alpha_1 \alpha_2 \hat{c} + (1 - \alpha_1 \alpha_2) w^y\big) \leq 1 \qquad \text{(resp. } \geq 1) \tag{3.6c}$$

Under Eqs. (3.6a)–(3.6c), "topological chaos" will occur for the dynamical system in Eq. (3.5).

Benhabib and Day considered other relevant economic issues pertaining to the model, such as the role of a central authority regulating the credit used by the young and the Pareto efficiency of the chaotic trajectories (they may well be such, under very general conditions). In a brief remark, they also addressed the Samuelson case, pointing out that, for the case in which cyclic or chaotic trajectories could obtain, the dynamical system of Eq.(3.5) would not be well defined, in the sense that for each c_t^y there will exist two equilibrium levels of c_{t+1}^y. In the absence of a convincing selection criterion, they saw no purpose of analyzing such a system.

This very same case (the Samuelson one) was instead taken up and worked out in all its details in a paper by Grandmont (1985). The length of this article prevents a description of all the results there obtained. I will content myself with a brief discussion of the basic technique used by the author to define a meaningful dynamical system for the Samuelson case and to prove that it can be chaotic.

The basic model is, as before, one with overlapping generations. Assume the utility function is time separable and, instead of a fixed endowment of consumption good, assume each agent has a labor-time endowment $\bar{\ell}^i$, where $i = y, o$, in each period of his life. Denote with ℓ^i the amount of $\bar{\ell}^i$ he supplies for work and assume his utility depends both on consumption c^i, where $i = y, o$, and leisure time $\bar{\ell}^i - \ell^i$, according to $U_1^y(c^y, \bar{\ell}^y - \ell^y) + U^o(c^o, \bar{\ell}^o - \ell^o)$, where the utility function U^i satisfy, for $i = y, o$, standard differentiability monotonicity and strict concavity hypotheses. As we want to consider the case in which the young lends to the old in exchange for "money," let us introduce the latter explicitly in a fixed amount M. The representative individual once again maximizes his utility subject to the budget constraints:

$$p_t(c_t^y - \ell_t^y) + m^d = 0 \tag{3.7a}$$

$$p_{t+1}^e(c_{t+1}^o - \ell_{t+1}^o = m^d \tag{3.7b}$$

Here m^d denotes the nominal amount of money demanded by the young. Notice that we assume that the technology to be such that a unit of labor is transformed into a unit of consumption good so that we are still facing a pure exchange economy. Note also that, for now, the perfect foresight assumption about future prices used by Benhabib and Day has not not been made and p_{t+1}^e denotes the expected future price as of time t. Such a maximization problem has, once again, a unique solution which will depend only on $\rho_t^e = p_t/p_{t+1}^e$, the expected interest factor. By modifying slightly the notation used previously, we can define an excess demand for the good $z^i(\rho^e)$, where $i = y, o$, as $z^i(\rho^e) = c^i - \ell^i$. Remember that we are considering the case in which the young lends to the old in exchange for money. This implies that the $z^y(\rho^e)$ will always be negative and such that $m^d = M = -p_t z^y(\rho_t^e) = p_{t+1}^e z^o(\rho_t^e)$ at each t, along an equilibrium path. Conversely, when old, each agent will spend all of his money stock in exchange for goods. In the equality above, M denotes the fixed amount of existing bills that must all be demanded by the young in equilibrium.

Assume now that agents have perfect foresight, i.e., $p_{t+1}^e = p_{t+1}$. From the equilibrium conditions given above and the material balance, it follows that a competitive equilibrium is a sequence of p_t that solves

$$z^y(\rho_t) + z^o(\rho_{t-1}) = 0 \tag{3.8a}$$

$$p_{t+1} z^o(\rho_t) = M \tag{3.8b}$$

Notice that what matters for the dynamic is Eq. (3.8a); once the sequence of ρ_t is determined we will get the price level from Eq. (3.8b) as a function of the given M. Our system satisfies the Quantity Theory of Money, the latter is no obstacle to chaos. If we try to invert z^y to obtain a "forward dynamics" (i.e., ρ_t as a function of ρ_{t-1}), we get into the problem recalled above as z^y maybe backward bending. This is the crucial feature of the model as well as the main source of the aperiodic behavior. An increase in ρ_t has two conflicting effects on the demand for consumption by the young. It has an intertemporal substitution effect as it makes consumption more expensive today (hence, z^y should increase). But this is not the case or z^o. A little thought along the same lines will convince the reader that when ρ_{t-1} goes up, both substitution and wealth effect will push up the old agent's demand for consumption. Therefore z^o may be inverted and a (fictitious) backward dynamics can be obtained from Eq. (3.8a):

$$\rho_{t-1} = \phi(\rho_t) \equiv (z^o)^{-1}\big(-z^y(\rho_t)\big). \tag{3.9}$$

Even if the "trick" behind Eq. (3.9) is not quite the full solution to our problem, it suggests the line along which a perfect foresight dynamics can be studied for an economy of the Samuelson type. Conscious of this fact, Grandmont dedicates a large part of his paper to clarify the relation between the backward and the forward dynamics, as well as to work out the implications that different expectation-formation rules have for the stability of the system (on this point, see also Grandmont and Laroque, 1986). We have to skip all this for reasons of brevity. What matters to us is that, given a periodic trajectory for the backward dynamics, one may generically define a forward dynamics in a neighborhood of such trajectory so that a stability analysis can be conducted by reversing the dynamic properties of the backward paths. By following this strategy the author gives conditions under which Eq. (3.9) defines a dynamical system as an iterated map of an interval into itself and carries on a complete bifurcation analysis of such system. He makes abundant use of the techniques and results illustrated in Collet and Eckmann (1980) in order to show that a period-doubling bifurcation cascade leading to chaos will originate for a large class of utility functions. In particular, it turns out that when z^y is not monotonic (for the reasons given above), then a large enough degree of risk aversion on the part of the old trader (i.e., a "very concave" U^o) will lead to chaos if certain technical conditions are satisfied.

A more detailed consideration of his results would show that they confirm and generalize the earlier proof given by Benhabib and Day for the classical case. Chaos originates out of a conflict between the wealth and intertemporal substitution effects created by a variation in the real interest rate if the first effect is strong enough. Finally, it is worth noting that, contrary to many of the papers considered in this and the next section, such paths are not necessarily Pareto efficient and the cycles may be dampened (or created) by appropriate monetary and fiscal policies.

It is not exaggerating to say that Grandmont's paper has had a much bigger impact on the economics profession than any other of the previous (and, for that matter, subsequent) works on chaos in economics. It is after this work that macroeconomists and economic theorists in general have started to realize that, indeed, there may be something in an endogenous theory of business cycles that cannot be captured by the prevailing linear-stochastic approach.

This new attention has made it possible to better reconsider some of the former studies I have been illustrating here as well as provide incentives for new research on different models. As far as the overlapping generations economies are concerned, some recent efforts have generalized and improved upon the older results. Reasons of space suggest brevity; therefore, I will only sketch some of these discoveries. Farmer (1986) has considered a variation of the basic model where capital is introduced both as a means of production and as an asset: it is proven that, when government debt is present to finance a deficit of fixed value, periodic orbits may be obtained for the two-dimensional discrete time system that describes the economy's evolution. The technique used here is that of Hopf bifurcation for maps on the plane. The role of production is more fully analyzed in Reichlin (1986; see also Reichlin, 1987, for further improvements along the same line). In particular, the author is able to show that, when a nontrivial technology is present, one does not need the empirically unlikely assumption made by Grandmont which requires saving to be a decreasing function of the interest rate when the latter is high enough (strong wealth effect) in order to obtain complicated dynamic behaviors. In fact, by means of simple production functions (either fixed coefficients or CES) that use labor and the invested amount of the homogeneous good to produce new output, Reichlin obtains a dynamical system for the capital stock which is represented by a map of the plane into itself. He also uses the Hopf theorem to prove the existence of a limit cycle. The result is obtained even if saving is a monotonically increasing function of the rate of interest, as long as the elasticity of substitution between factors of production is low enough. This is consistent with a result obtained by Boldrin (1986; see Section 4 below) for an economy with infinitely living agents, where an example with either a CES or Leontief production function exhibits chaos when the elasticity of substitution for the CES is low, even if agents do not have a high discount factor for future utilities. In Reichlin (1987), the same overlapping generations economy is considered with a two-sector technology. In this case, the author is able to show the existence of chaotic trajectories. Finally, Aiyagari (1987) proves the existence of periodic orbits for an exchange economy with overlapping generations that do not live only for two periods but for finitely many ones.

4 Economies with A Finite Number of Infinitely Lived Agents

There are many reasons for which one may feel unsatisfied with the type of world described in an OLG model. From the dynamic point of view, which concerns us here, these models appear rather farfetched. Each time period has to be interpreted, empirically, as equivalent to 30–40 years which makes it impossible to define observable counterparts for the variables of the model. On the other hand, each agent behaves very myopically as nobody cares for the consequences of his actions more than one period ahead. This seems intuitively at odds with the existence of many institutions (firms, *in primis*) that participate in the markets for very long period of time and that should therefore try to forecast the implications of their choices for the far future. Finally, it is reasonable to claim that a "good government" should be one which takes into account the interests of all the generations, even of the unborn, in pursuing its policies and this fact should lead to programming over infinite horizons of time. This was, indeed, Frank Ramsey's concern when, in the 1920's, he first proposed to consider optimal programs that maximize an infinite sum of society's welfares from the initial period up to infinity.

To make a long story short, such basic intuitions have led many scholars to consider the behavior of competitive economies where a finite number of agents live forever and, being endowed with perfect foresight, try to maximize the discounted sum of their utilities over the infinite horizon. The literature on this field is enormous; the curious reader is referred to Arrow and Kurz (1970), Cass and Shell (1976), Bewley (1982) and especially McKenzie (1986, 1987) for more complete treatments. For our purposes it suffices to sketch here the basic ingredients of a very general model from which most of the adopted setups can be derived as special cases. In particular, we will consider a world with a single (representative) agent that controls both consumption and production decisions and perfectly foresees even the more distant future (see Bewley, 1982, and the literature therein for a reconciliation of this abstraction with the case of many independent consumers and producers). Also we will describe only the discrete time formalism even, if, later on, we will have to sue the continuous-time version of the same model; the translation should be immediate.

In every period $t = 0, 1, 2, \ldots$, an agent derives satisfaction from a "consumption" vector $c + t \in I\!R^m$, according to a utility function $u(c_t)$ which is taken increasing, concave and smooth as needed. Notice that c_t denotes a flow of goods that are consumed in period t. The state of the world is fully described by a vector $x_t \in I\!R^n_+$ of stocks and by a feasible set $F \subset I\!R^{2n}_+ \times I\!R^m$ composed of all the triples of today's stocks, today's consumptions and tomorrow's stocks that are technologically compatible, i.e., a point in F has the form (x_t, c_t, x_{t+1}). Now define

$$V(x, y) = \max_c u(c) \quad \text{s.t.} \quad (x, c, y) \in F \tag{4.1}$$

and let $D \subset I\!\!R^{2n}_{+}$ be the projection of F along the c's coordinates. Then V, which is called the short-run or instantaneous return function, will give the maximum utility achievable at time t if the state is x and we have chosen to go into state y by tomorrow. It should be easy to see that to maximize the discounted sum $\sum_{t=0}^{\infty} u(c_t)\delta^t$ s.t. $(x_t, c_t, x_{t+1}) \in F$ is equivalent to $\max \sum_{t=0}^{\infty} V(x_t, x_{t+1})\delta^t$ s.t. $(x_t, x_{t+1}) \in D$.

The parameter δ indicates the rate at which future utilities are discounted from today's standpoint (impatience): it takes values in $[0, 1)$. For $\delta = 0$ the agent is infinitely impatient and there is a sense in which a repeated myopic optimization of this kind may represent the outcomes of an OLG model. In general δ will be greater than zero.

It is mathematically simpler to consider the problem in the latter (reduced) form. The following assumptions on V and D may be derived from the more basic hypotheses on u and F

Assumption 1. $V\colon D \to I\!\!R$ is strictly concave and smooth (if needed). $V(x, y)$ is increasing in x and decreasing in y.

Assumption 2. $D \subset X \times X \subset I\!\!R^{2n}_{+}$ is convex, compact and with non-empty interior. X is also convex, compact and with non-empty interior.

The initial state x_0 is given. Notice that the economy we are describing is essentially time-invariant: return function and feasible set do not change over time, the latter enters the picture only through discounting, and the intrinsically intertemporal nature of the production process is summarized by D (see McKenzie, 1986, for the case in which V and D evolve exogenously with time in a fairly restricted way; very little can be said about this case).

The optimization problem we are facing can be equivalently described as one of Dynamic Programming:

$$W(x) = \max \left\{ V(x, y) + \delta W(y), \quad \text{s.t.} \quad (x, y) \in D \right\} \tag{4.2}$$

The latter is the Bellman equation and $W(x)$ is the value function for such a problem. A solution to Eq. (4.2) will be a map $\tau_\delta \colon X \to X$ describing the optimal sequence of states $\{x_0, x_1, x_2, \dots\}$ as a dynamical system $x_{t+1} = \tau_\delta(x_t)$ on X. The time evolution described by τ_δ contains all the relevant information about the dynamic behavior of our model economy. In particular, the price vectors p_t of the stocks x_t that realize the optimal program as a competitive equilibrium over time follows a dynamic process that (when the solution $\{x_t\}$ is interior to X) is homeomorphic to the one for the stocks. In other words, $p_{t+1} = \theta(p_t)$ with $\theta = \delta DW \cdot \tau \cdot (DW\delta)^{-1}$, where D is the derivative operator.

The question that concerns us is: what are the predictions of the theory about asymptotic behavior of the dynamical system τ_δ? Where would a stationary economy converge under Competitive Equilibrium and Perfect Foresight? A first, remarkable answer is given by the following:

Turnpike Theorem: Under Assumptions 1 and 2, ther exists a level $\bar{\delta}$ of the discount factor such that for all the δ's in the non-empty interval $[\bar{\delta}, 1)$, the function τ_δ that solves Eq. (4.2) has a unique globally attractive fixed point $x^* = \tau_\delta(x^*)$. Such an x^* is also interior to X under additional mold restrictions.

Not too bad indeed. Under a set of hypotheses as general as Assumptions 1 and 2, we are able to predict that if people are not "too impatient" relative to the given V and D, then they should move toward a stationary state where history repeats itself indefinitely and no surprises ever arise. In the form given here, the Turnpike Theorem is due to Scheinkman (1976), whereas McKenzie (1976) and Rockafellar (1976) proved it for the continuous-time version; Bewley (1982) and Yano (1984) generalized it to the many-agents case (but see McKenzie, 1986, for a more careful attribution of credits).

As remarkable as it is, the Turnpike property is also very sensitive to perturbations of its sufficient conditions. In particular, how close should $\bar{\delta}$ be to one in order to obtain convergence and what happens when δ is smaller than $\bar{\delta}$? These are important questions. It is hard to rely heavily on a property that may depend critically on such a volatile and unobservable factor as "society's average degree of impatience."

The careful reader should have realized by now that the one-sector model we briefly introduced at the end of Section 2, and used by Dechert to prove that cycles and chaos are not optimal in that framework, is a special case of the general model we are considering here, with $V(x_t, x_{t+1}) = u\big[f(x_t) - x_{t+1}\big]$ and $D = \big\{(x_t, x_{t+1}) \text{ s.t. } 0 \leq x_{t+1} \leq f(x_t)\big\}$. For that model, the Turnpike Theorem holds independently of the discount factor as τ_δ s always monotonically increasing. Unfortunately, such a nice feature does not persist even if the simplest generalization of the one-sector model is taken into account. This was proved by Benhabib and Nishimura (1985). They considered a model with two goods—consumption and capital—which are produced by two different sectors by means of capital and labor. Given the two production functions, one can define a Production Possibility Frontier (PPF) $T(x_t, x_{t+1}) = c_t$, that gives the producible amount of consumption when the aggregated capital stock is x_t (a scalar), labor is efficiently and fully employed, and the decision of having an aggregated stock x_{t+1} tomorrow has been taken. The return function is now $V(x_t, x_{t+1}) = u\big[T(x_t, x_{t+1})\big]$ and $D = \big\{(x_t, x_{t+1}) \text{ s.t. } 0 \leq x_{t+1} \leq F(x_t, 1)\big\}$ where F is the production function of the capital good sector and labor has been normalized to one. In such a case, τ_δ is not always sloping upward. If the consumption sector uses a capital/labor ratio higher than the one used by the capital sector, it will slope downwards. Let x^* be the (unique) interior fixed point (i.e., $\tau_\delta(x^*) = x^*$). This is the candidate for the Turnpike. Assume, for simplicity, that τ_δ is differentiable in a neighborhood of x^*. The derivative will be $\tau'_\delta(x^*)$ at the steady state, it is negative, and it changes as δ moves in $(0, 1)$, everything else being equal. Benhabib and Nishimura showed that it may take on the value -1 for admissable δ's, in such a way that the conditions for a flip (period-doubling) bifurcation are realized. In this case, an optimal cycle of period-two will exist which can also be attractive: no more Turnpike! One may provide examples of this phenomenon showing that such an outcome is by no means due to "pathological" technologies and preferences.

Not only this, cycles are not a special feature of the discrete-time version of our model. In fact, in a much earlier work (see Benhabib and Nishimura, 1979), the two authors had used the Hopf bifurcation theorem to prove that limit cycles can occur in the continuous-time case. Let us show very briefly how this can happen. In continuous time, we face an optimal control problem of the form:

$$\max \int_0^\infty V(x,\dot{x}) \exp(-\rho t) \quad \text{s.t.} \quad (x,\dot{x}) \in D, \quad x(0) \text{ given.} \tag{4.3}$$

Here $x(t)$ is a vector depending on time, $\dot{x}$ is its time derivative, D again the convex feasible set, and ρ the discount factor in $[0,\infty)$ ($\rho = 0$ is equivalent to $\delta = 1$ in discrete time). Using the Maximum Principle, one defines a Hamiltonian:

$$H(x,q) = \max_{\dot{x}} \left\{ V(x,\dot{x}) + < q, \dot{x} >, \quad \text{s.t.} \quad (x,\dot{x}) \in D \right\} \tag{4.4}$$

which can be interpreted as the current value of national income evaluated at the (shadow) prices q (on this point see Cass and Shell, 1976).

The dynamical equation is then

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,q)}{\partial q} \\ \dot{q} &= \frac{-\partial H(x,q)}{\partial x} + \rho q. \end{aligned} \tag{4.5}$$

Linearization of Eq. (4.5) around the steady state will yield, after some manipulations, a Jacobian matrix J that can be written as $J = \tilde{J}(\rho/2)I.$, where I is the $2n \times 2n$ identity matrix. As $\tilde{J}$ is a Hamiltonian matrix, we may consider how its eigenvalues will change with the discount factor ρ and then add $\rho/2$ to obtain those of J. If $\rho = 0$, $\tilde{J}$ has the form

$$\tilde{J} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \tag{4.6}$$

with $A = \partial H(x,q)/\partial x \partial q$, $B = \partial^2 H(x,q)/\partial^2 q$, and $C = -\partial^2 H(x,q)/\partial^2 x$. It is a result of Rockafellar (1973) that under strict concavity in x and strict convexity in q of H, the $2n$ eigenvalues of $\tilde{J}$ will split into n positive and n negative ones. The steady state will be a saddle point with a stable manifold of dimension n. As the latter is also the dimension of the control vector, the optimal program will steer the system on the stable manifold, thereby guaranteeing convergence to the Turnpike. For $\rho > 0$, this is not necessarily true: the saddle-point property may be lost as some of the negative eigenvalues become positive. The Turnpike Theorems give conditions under which such stability property is preserved for small ρ. But as Benhabib and Nishimura showed when ρ grows, a pair (or more than a pair) of eigenvalues may change the sign of their real part by crossing the imaginary axis. In such a case, they proved that (care taken for the technical details) a Hopf bifurcation is realized. The limit cycle associated with it may indeed be an attractor for the system of Eq. (4.5). Once again the Turnpike property is lost as people become "a bit more impatient" than the economists would like!

Some characteristics of the oscillatory paths so obtained need to be stressed. First of all, they are realized as "equilibrium paths," in the sense that all markets are continuously clearing at each point in time, prices adjust completely and no productive resource is left "involuntarily unemployed." Moreover, they are Pareto efficient in the sense that it is impossible to modify the allocation of resources they imply, in order to increase the welfare of some agent without making somebody else worse off. Let us make it clear that none of the authors working along these lines seems to imply, with this, that economic fluctuations are intrinsically good and that nothing can or should be done to modify and control them. The idea instead is of showing that there exist forces that are intrinsic to the competitive mechanism, and depend upon the technological structure of the economy, that can be a source of wide oscillations for output and prices. As it was the case with the discrete-time two-sector model, it is the existence of certain factor-intensity relations across sectors that make it profitable for the producers (and the consumers alike) to invest, produce (and consume) in an oscillatory form. Even if all the prices are the "right ones" (i.e., no conditions for profitable arbitrage exist), still the pure seeking of individual profits will bring about cyclic behavior.

It is opportune to admit that we do not presently have an empirical representation for this phenomena and that our ability to measure how intersectoral profitability relations affect the cycle is very small or almost nil. Nevertheless, they follow from sound economic theory and it is hard to rule them out on pure *a priori* grounds. As we will see in a moment, this very same logic can be pursued further to explain the origin of chaotic movements for the same class of model economies.

Indeed, the possibility of more complicated trajectories had already been envisaged by Benhabib and Nishimura (1979, p. 433), where they quote Ruelle and Takens' celebrated paper on turbulence, after noticing that further bifurcations may follow the Hopf one, giving rise to a torus, etc. It nevertheless took a few years before a full proof was produced, and when it came it was more general than expected. Every dynamics turned out to be a possible solution for an economy satisfying Assumptions 1 and 2 above. This was proven in Boldrin and Montrucchio (1986b; but see also Boldrin and Montrucchio, 1984 and 1986a, and Montrucchio, 1986, for additional results). The result, formally speaking, has the following form: let $\theta\colon X \to X$ be a C^2-map describing a dynamical system on the compact convex set $X \subset I\!R^n$; then there exist a technological set D, return function v, and a discount factor $\delta \in (0,1)$ satisfying Assumptions 1 and 2, such that θ is the policy function τ_δ that solves Eq. (4.2) for a given D, V and δ. The proof given was a constructive one, so that one may effectively compute a fictitious economy for each given dynamics. This clarifies that any kind of strange dynamic behavior is fully compatible with competitive markets, perfect foresight, decreasing returns, etc. At about the same time, Deneckere and Pelikan (1986) also presented some one-dimensional examples of models satisfying our assumptions and having the quadratic map $4x(1-x)$ as their optimal policy function.

All these results were given for the discrete-time version of the model, but they are not specific to it. In Montrucchio (1987), it in fact has been proven that exactly the same results hold for the case in which time is continuous. One big question that still remains open in this area pertains to the economic logic behind these theoretical and mathematical results. What is it that makes it profitable for a competitive economy to oscillate erratically over time? As we noticed with respect to the works of Benhabib and Nishimura on limit cycles., the driving force seems to be the technological structure of the different sectors. A very similar answer is true for the chaotic motions.

Unfortunately, we do not have a full-fledged analytical explanation for the multisectoral case but something can be said for the two-sector, two-good economy that is often used in macroeconomic applications. A theoretical analysis is provided in Boldrin (1986). There are two goods—consumption and capital—produced by means of two factors—capital itself and labor. The model is therefore the same as in Benhabib and Nishimura (1985) and the resulting dynamics in the aggregate capital stock k_t is one-dimensional. It is shown that the policy function $k_{t+1} = \tau_\delta(k_t)$ is unimodal when factor-intensity reversal occurs between the two sectors. Remember that for the case in which the consumption sector always uses a capital-labor ratio higher than the one of the capital sector, period-two cycles ae possible. More often than not, it is possible to find a level, say, k^*, of the aggregate capital stock such that when k_t is in $[0, k^*)$, the capital sector has a higher capital-labor ratio, whereas the opposite happens when k_t is in $(k^*, \bar{k}]$, where $\bar{k}$ is the maximum level of capital that the economy can sustain. This technological feature provides the unimodal shape for τ_δ. Variations in the level of the discount factor δ then can produce a cascade of period-doubling bifurcations that (technicalities aside) leads to period-three orbits and even to chaos in the sense of the existence of an invariant and absolutely continuous ergodic measure.

A simple example that uses standard production functions is also provided in the same paper. The problem is taken up again in Boldrin and Deneckere (1987). The case of a two-sector economy with a Cobb-Doublas and a Leontief production function is here studied in detail. That such an economy would have chaotic trajectories was first conjectured by José Scheinkman in Scheinkman (1984). Boldrin and Deneckere provide a full proof for this assertion and also show how, depending on the various parameters, cycles of different lengths can be organized.

A by-product of these exercises is to make clear that the often-adopted criticism that asserts the irrelevance of this approach to business cycle theory because of the "too high" level of discounting required for chaos is, indeed, wrong, Clearly a level of δ too close to zero as in the earlier examples would imply annual "interest rates" of the 1000% magnitude. But this need not occur. In particular, the example provided in Boldrin (1986) shows that δ may be very close to one (say, in the range of $(0.6, 0.8)$) and chaos may still be possible for appropriate values of the technological parameters.

The open question is this: can we move from the "toy models" stage to a fully specificed, disaggregated and empirically calibrated model of the same class that will reproduce the so-called "stylized facts" of observed business cycles? The question is open.

This section would be incomplete without a reference to the innovative work that Mike Woodford has conducted in this area. The best example of it is provided by Woodford (1987; but see other references to the same author therein). What he uses is the one-sector growth model we described at the end of Section 2.3. We recalled there the result of Dechert (1984), according to which only monotonically convergent orbits are possible in such a setup. Woodford asks the simple question: what happens if, for some reason, one of the agents is not free to borrow? That is to say, what if (as in the real world) there are no markets open for trade at very distant dates, or loan markets for financing investments above a certain amount are not available? This intuition is captured by postulating two different types of agents—consumers and entrepreneurs. The latter are in charge of the investment process, but they cannot borrow against future returns. They must finance their investments only out of funds generated internally to the firms. The author provides very good arguments for such a state of affairs to occur (idiosyncratic shocks and private information about returns, for example) and also shows that, in such a case, both equilibrium cycles and chaotic equilibrium dynamics may exist under very general hypotheses about the technology and the level of discounting. The latter, in particular, may be as low as one likes, without affecting the result.

Considering the appealing aspect of the hypothesis on the lack of complete markets, it seems to me that such a line of research appears as one of the most promising, and worth pursuing further.

5 Conclusions

It is difficult to conclude, when a research topic is as recent, open and chaotic as the one we have surveyed. The real issue is: where do we go from here?

It seems clear that chaotic behaviors are not rare, at least theoretically, in well-formulated economic contests. In fact, they seem to be pervasive of even the simplest, descriptive representations of economic dynamics, as we have tried to show in Section 2.

When the discipline of utility maximization and rational intertemporal choice is imposed on the behavioral rules, they do not disappear at all. The research effort conducted so far has been able to identify classes of economic factors that may explain such persistence: (a) the relative importance of wealth effects as opposed to intertemporal substitution effects, (b) the degree of factor substitutability in production and the different factor-utilization ratios across sectors; (c) the degree of people's impatience and/or the extent to which they behave myopically with respect to future events; and (d) the lack of certain markets, especially of those for borrowing-lending against expected future returns.

It is difficult to establish an order of importance among these elements and they need not exhaust the class of possible explanations. What matters is that they all make sense from the point of view of economic theory; the endogenous approach to economic fluctuations appears therefore well grounded within the established General Equilibrium paradigm.

How far we go with this, on the practical side, is not clear. We are still in the stage of very abstract, purely qualitative models: their predictions are so general and so vague that any hope of testing them directly will be easily frustrated. On the other side, the empirically oriented works surveyed by Brock in his contribution to this volume suggest that there are important results we may achieve along those lines.

What we need, therefore, is to construct models that can be parameterized by using empirical evidence and that can yield testable, even if primitive, predictions by means of computer simulations. I believe it would be most useful, at the present stage, to channel our research along these lines.

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