Learning-By-Doing, International Trade and Growth: A Note

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1 Introduction

The research effort that underlies the simple model we are presenting was motivated by a few (to us) undisputable facts. The first, and most evident, is that across the world different countries have been growing at very different speeds during, say, the last fifty years. In particular, if we want to interpret their behavior in terms of steady-state growth rates, we have to conclude that they are on different steady states: some countries grow very fast, some others at a slower pace, and a few do not seem to grow at all. A second fact is, maybe, less "theory free" but, in our opinion, equally compelling: such differences in the rate of development do not seem to be explainable in terms of differences in natural resources, capital stocks, technologies and tastes. In particular, if we define "technology" as a list of available blueprints describing how to combine inputs to obtain outputs and "labor force" as some measure of the existing population, then it should be easy to see that, with any suitable definitions of taste and natural resources, there exist countries that are similar in any respect (or at least were similar when their development processes started), but that have been growing very differently. An easy way out is always available: to claim that tastes are indeed different, that some countries are inhabited by people with a disutility of work and/or high discount rate so that they do not work, do not save and consequently do not grow. But this seems nothing more than a trick.

Finally, it is also a fact that countries growing at different rates end up producing distinct sets of goods: they may or may not completely specialize, but it is certain that the product mix of the fast-growing nations will typically contain a larger portion of high-technology, advanced, non-primary goods than the one of the slow-growing countries. In short, if we aggregate goods in "low tech" and "high tech," then the process of development seems to imply a specialization in the second group for those countries that exhibit high rates of growth. Our question is: can we build a model that accommodates the three qualitative facts listed above and that does it in a parsimonious way, i.e., without introducing a plethora of special assumptions on preferences, market structures, trading constraints, etc.? The answer is positive, at least to a first approximation.

We consider a world with two countries, two produced goods and a finite number of inputs, exogenously supplied in fixed quantities. The consumers in each country are assumed to satisfy the standard neoclassical hypothesis and the production of each good is organized competitively in the presence of many identical firms. The output of each single firm in an industry is a function of the amounts of inputs it hires and of the average level of "expertise" in the country. The amount of the latter factor that we denote by θ, may be increased only through learning-by-doing (see Arrow, 1962). More precisely, the rate of growth of "expertise" in a country depends on the share of its work force allocated to the production of each good. We specify that the first good is a "high technology" (industrial) product whereas the second one is a "low technology" (agricultural) commodity. It is then natural to assume that a certain effort allocated to the production of the industrial good will have a larger positive effect on the growth rate of θ than the same amount allocated to production in the agricultural good. The idea here is that by producing potatoes, one may get some increase in overall expertise, but not as much as when producing computers. We also assume that as θ increases, its productivity in the industrial sector increases relative to the agricultural sector.

Finally, we assume that except perhaps for the initial values of θ, the two countries are identical.

At each point in time, a competitive firm takes as given prices and its production possibilities in making its input-output decisions. Each firm's decision, in turn, affects the future production possibilities of all firms in the same country, but given the presence of many producers in a country, the individual firms correctly ignore the impact of their decision on their own future production costs. It is the presence of this *externality* that makes our model not entirely conventional.

It is clear that in such a framework, a small difference in the initial levels of θ may be magnified by the dynamical process. In fact, the country with larger θ at the beginning will have some comparative advantage in producing the industrial good which in turn will reinforce such advantage as the learning-by-doing mechanism is stronger in this sector. We will observe then two different growth rates in the two countries and, if some steady state exists to which they converge, it will be an asymmetric position in which one country is richer than the other (i.e., has a higher level of θ), produces a larger proportion of the industrial commodity, and pays its factors of production higher returns as their marginal productivity is in fact higher in the rich than in the poor country. All of this simply follows in competitive equilibrium as a consequence of the initial difference in expertise, everything else being identical.

The remark that externalities may affect the dynamic evolution of comparative advantages was previously made by Krugman (1985) and Lucas (1985). Both dealt with a linear technology and with industry-specific knowledge. As a consequence, the results they obtained are similar to the ones in Section 3 below.

Finally, in contrast with Krugman (1985) and Lucas (1985) our learning-by-doing is not an industry-specific mechanism, i.e., the variable θ measures the overall level of expertise for the country as a whole. We have made this choice partly because we believe that this type of externality actually spills over across industries and partly because it leads to simpler mathematics without any loss of explanatory power.

One may also argue that a growth mechanism driven only by learning-by-doing does not look very attractive. In particular, the assumption that all of the "expertise/technical knowledge" is disembodied seems to be rather odd. We believe that this is a serious issue to which a more detailed analysis should be dedicated. It is our conjecture that the appropriate route is to embody the advancement in expertise and/or knowledge in the capital goods and allowing accumulation of such goods. The embodiment may or may not be full, as human capital and pure expertise factors ought to be considered. Nevertheless, we believe capital accumulation to be an essential instrument through which progress in productive ability and efficiency of an economic system are transferred over time. The rest of the present note is organized into three other sections and some brief conclusions. In the next section, we present a general formalized version of the world economy we have in mind, solve for a competitive equilibrium, and briefly describe the dynamic process for expertise and the associated competitive growth path. In the third section, we present a simple linear model where such a dynamic is realized, albeit in a very extreme form. Finally, Section 4 contains an analysis of the conditions under which the general model of Section 2 produces the asymmetric outcomes we have in mind.

2 The General Framework

Consider a world with two countries and two goods: let $i = 1, 2$ denote the countries, and x and y denote the goods. Each country is inhabited by a large number of identical, infinite-lived agents, maximizing their lifetime discounted utility from consumption and supplying a fixed quantity of labor in each period. The latter together with a finite number of productive resources that are inelastically supplied in fixed quantities during each period, will be denoted by the vector z^i. Let θ^i denote the quantity of "expertise" in country i. Notice that all variables depend on time, but we suppress the t-variables as we are using a continuous time setup.

2.1 Technology

At each time t, the firms in country i have the production functions

$$x^i = \alpha_x(\theta^i)F_x(z_x^i) \tag{2.1a}$$

in the x-producing industry, and

$$y^i = \alpha_y(\theta^i)F_y(z_y^i) \tag{2.1b}$$

in the y-producing industry. Here z_j^i, for $j = x, y$, denotes the amount of inputs employed in sector j, with $z_x^i + z_y^i = z^i$, constant over time.

We assume:

(T.1) *For $j = x, y$, $F_j\colon \mathbb{R}_+^m \to \mathbb{R}_+$ is an increasing, homogeneous degree-one and concave function which is C^2 on the interior of its domain.*

Note that under (T.1), Eqs. (2.1a) and (2.1b) also define the industry's production function.

We also assume:

(T.2) *$\alpha_j\colon \mathbb{R}_+ \to \mathbb{R}_+$, for $j = x, y$, is smooth almost everywhere on $\mathbb{R}_+$ and $\lim_{\theta\to\infty} \alpha_j(\theta) \leq A_j$, where A_j is a finite number. Also α_x is strictly increasing and α_y non-decreasing.*

2.2 Preferences

The representative agent in each country maximizes his period-by-period utility function $u(c_x^i, c_y^i)$, which amounts to intertemporal maximization as no savings are allowed and the learning-by-doing mechanism works as a pure external effect. Of his utility function we assume:

(U.1) *$u\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ is strictly concave, homothetic and of class C^2. Also it satisfies*

$$\lim_{c_j\to 0} \frac{\partial u(c_x, c_y)}{\partial c_j} = +\infty, \qquad j = x, y.$$

Consumers in country i own the total amount of resources z^i and lend them out to the firms at a price π^i which will be, in equilibrium, a function of $(\theta^1, \theta^2) = \theta$. Write $M^i(\theta)$ for the income that they so receive and $P(\theta)$ for the price of the good x in terms of the good y. By solving the problems

$$\max u(c^i_x, c^i_y), \qquad \text{s. t. } c^i_y + P(\theta)c^i_x \le M^i(\theta) \tag{2.2}$$

for $i = 1, 2$ we get the four demand functions

$$c^i_j = d_j\big(P(\theta), M^i(\theta)\big), \qquad i = 1, 2; \quad j = x, y. \tag{2.3}$$

2.3 Competitive Equilibrium

On the supply side, for given $\theta = (\theta^1, \theta^2)$, each country allocates z^i competitively across sectors, taking $P(\theta)$ and π^i as given. The maximum problems that are solved are:

$$\max\colon P(\theta)x^i - \langle z^i_x, \pi^i(\theta)\rangle \qquad \text{s. t. } x^i \le \alpha_x(\theta^i)F_x(z^i_x) \tag{2.4a}$$

$$\max\colon y^i - \langle z^i_y, \pi^i(\theta)\rangle \qquad \text{s. t. } y^i \le \alpha_y(\theta^i)F_y(z^i_y) \tag{2.4b}$$

for $i = 1, 2$.

Once again we will have a vector of factor-demand correspondences in each country and related supply correspondences for the output that will depend, parametrically, on θ:

$$z^i_x \in z_x\big(P(\theta), \pi^i(\theta)\big) \tag{2.5a}$$

$$z^i_y \in z_y\big(P(\theta), \pi^i(\theta)\big) \tag{2.5b}$$

$$x^i \in \alpha_x(\theta^i)F_x\big[z_x\big(P(\theta), \pi^i(\theta)\big)\big] \tag{2.6a}$$

$$y^i \in \alpha_y(\theta^i)F_y\big[z_y\big(P(\theta), \pi^i(\theta)\big)\big] \tag{2.6b}$$

for $i = 1, 2$.

A competitive equilibrium at a certain time (for given θ) in the world economy is then defined by price functions: $\{P(\theta), \pi^1(\theta), \pi^2(\theta)\}$ such that

$$\begin{aligned} & d_x\big(P(\theta), M^1(\theta)\big) + d_x\big(P(\theta), M^2(\theta)\big) \\ & \qquad \in \big\{\alpha_x(\theta^1)F_x\big[z_x\big(P(\theta), \pi^1(\theta)\big)\big] + \alpha_x(\theta^2)F_x\big[z_x\big(P(\theta), \pi^2(\theta)\big)\big]\big\}, \end{aligned} \tag{2.7a}$$

$$\begin{aligned} & d_y\big(P(\theta), M^1(\theta)\big) + d_y\big(P(\theta), M^2(\theta)\big) \\ & \qquad \in \big\{\alpha_y(\theta^1)F_y\big[z_y\big(P(\theta), \pi^1(\theta)\big)\big] + \alpha_y(\theta^2)F_y\big[z_y\big(P(\theta), \pi^2(\theta)\big)\big]\big\}, \end{aligned} \tag{2.7b}$$

$$z^1 \in \big\{z_x\big(P(\theta), \pi^1(\theta)\big) + z_y\big(P(\theta), \pi^1(\theta)\big)\big\}, \tag{2.7c}$$

$$z^2 \in \big\{z_x\big(P(\theta), \pi^2(\theta)\big) + z_y\big(P(\theta), \pi^2(\theta)\big)\big\}, \tag{2.7d}$$

Budget constraints and normalization of the price of y at one will make either Eq. (27a) or (27b) redundant. Existence of an equilibrium is a trivial result under our assumptions; uniqueness is also easy to prove as we are in fact facing a "Hicksian" economy (see Arrow-Hahn (1971), p. 220). In equilibrium, the quantities $x^1(\theta)$, $x^2(\theta)$, $y^1(\theta)$, and $y^2(\theta)$ of goods will be produced and consumed in the two countries.

We remark that at each time t, θ^1, θ^2 are fixed and the competitive equilibrium described above will be "instantaneously" Pareto optimal, i.e., it will maximize the welfare at time t of the representative consumer subject to the production possibilities. All deviations from Pareto optimality are of a dynamic nature. Our learning-by-doing hypothesis states that the time variations of θ^i are determined by:

$$\dot{\theta}^1 = E\big(z_x^1(\theta), z_y^1(\theta)\big) \tag{2.8a}$$

$$\dot{\theta}^2 = E\big(z_x^2(\theta), z_y^2(\theta)\big) \tag{2.8b}$$

with E increasing in both of its arguments. What can be said about the dynamical system of Eqs. (2.8)? Given the level of generality we have kept so far, it seems highly improbable to prove anything specific about the patterns of evolution of our model economy. We try to show in Section 4 that, indeed, with a couple of additional assumptions, we may deduce a picture of the state space that fits with the one we have in mind. But, first, we like to turn to a simple example where the desired conclusions follow almost trivially.

3 The Skeleton of the Model: A Ricardian Economy

We begin by discussing a very simplified model that formalizes, in an extreme form, our basic intuition. We specify the two production functions to be linear in the exogenously supplied factor (labor) as in Lucas (1985, sect. V). Set:

$$x^i = \alpha_x(\theta^i)\ell_x^i \tag{3.1a}$$

$$y^i = \alpha_y(\theta^i)\ell_y^i \tag{3.1b}$$

Assume that only labor is used in production and normalize units so that $\ell_x^i + \ell_y^i = 1$ in both countries. For the sake of the example, let's take a logarithmic utility function in both countries:

$$u(c_x^i, c_y^i) = \beta \ell n c_x^i + (1-\beta)\ell n c_y^i \tag{3.2}$$

with $\beta \in (0,1)$. This will give, upon maximization under budget constraint, the demand functions:

$$c_x^i = \frac{\beta M^i}{P_x} \tag{3.3a}$$

$$c_y^i = \frac{(1-\beta)M^i}{P_y} \tag{3.3b}$$

where M^i is the total income of country i (at given θ), and P_x and P_y are the two prices. Denote with W^i the wage in country $i = 1, 2$ (this also will depend on θ). Maximization of profits on the part of the firms under the simple linear technology of Eqs. (3.1) yields the supply rules:

$$x^i(\theta^i, W^i, P_x) \begin{cases} = 0 & \text{if } \dfrac{W^i}{P_x} > \alpha_x(\theta^i) \\ \in (0,\infty) & \text{if } \dfrac{W^i}{P_x} = \alpha_x(\theta^i) \\ = \infty & \text{otherwise} \end{cases}, \tag{3.4a}$$

$$y^i(\theta^i, W^i, P_y) \begin{cases} = 0 & \text{if } \dfrac{W^i}{P_y} > \alpha_y(\theta^i) \\ \in (0,\infty) & \text{if } \dfrac{W^i}{P_y} = \alpha_y(\theta^i) \\ = \infty & \text{otherwise} \end{cases}. \tag{3.4b}$$

Labor market clearing will imply that in each country, the equilibrium wage will be:

$$W^i(\theta^i, P_x, P_y) = \max\{P_x \alpha_x(\theta^i), P_y \alpha_y(\theta^i)\}. \tag{3.5}$$

Remember that, in equilibrium, P_x and P_y also will depend on θ. It is clear from Eq. (3.5) that factor-price equalization does not need to hold in our model. The wages in the two countries will in general be different.

For given $\theta = (\theta^1, \theta^2)$ the instantaneous competitive equilibrium at time t is Pareto efficient even if (as noted in Section 2) the whole path described by such Competitive Equilibria is not a Social Optimum. In any case, for given θ, total income at time t for country i is

$$M^i(\theta^i, P_x, P_y) = \max\{P_x \alpha_x(\theta^i), P_y \alpha_y(\theta^i)\}. \tag{3.6}$$

The competitive equilibrium prices and quantities can finally be found by solving the international market-clearing conditions:

$$\frac{\beta}{P_x}[M^1(\theta^1, P_x, P_y) + M^2(\theta^2, P_x, P_y)] = x^1(\theta^1, W^1, P_x) + x^2(\theta^2, W^2, P_x) \tag{3.7a}$$

$$\frac{1-\beta}{P_y}[M^1(\theta^1, P_x, P_y) + M^2(\theta^2, P_x, P_y)] = y^1(\theta^1, W^1, P_y) + y^2(\theta^2, W^2, P_y) \tag{3.7b}$$

where Eqs. (3.4), (3.5) and (3.6) have to be used.

In order to describe the time evolution induced by the solution of Eqs. (3.7), we need to specify the learning-by-doing mechanism. Assume it is:

$$\dot{\theta}^i = f(\ell_x^i) + g(\ell_y^i) - \gamma^{\theta_i} \tag{3.8}$$

where $f \geq 0$, $f' > 0$, $g \geq 0$, $g' > 0$ and bounded above, and $\gamma > 0$.

To simplify the discussion, we have excluded intersectoral influences. The "depreciation" factor γ may raise some doubts; we claim that expertise and knowledge depreciates. People die or forget what they have learned, machines wear out and are destroyed, etc. It would be easier to argue this point if "expertise" was embodied in the factors of production, but, as we said, it is also very difficult and we prefer at this stage to settle for less. Let's consider Eqs. (3.7) and (3.8), and draw a phase-plan for the dynamical system of Eq. (3.8) in the (θ^1, θ^2) space (see Figure 1).

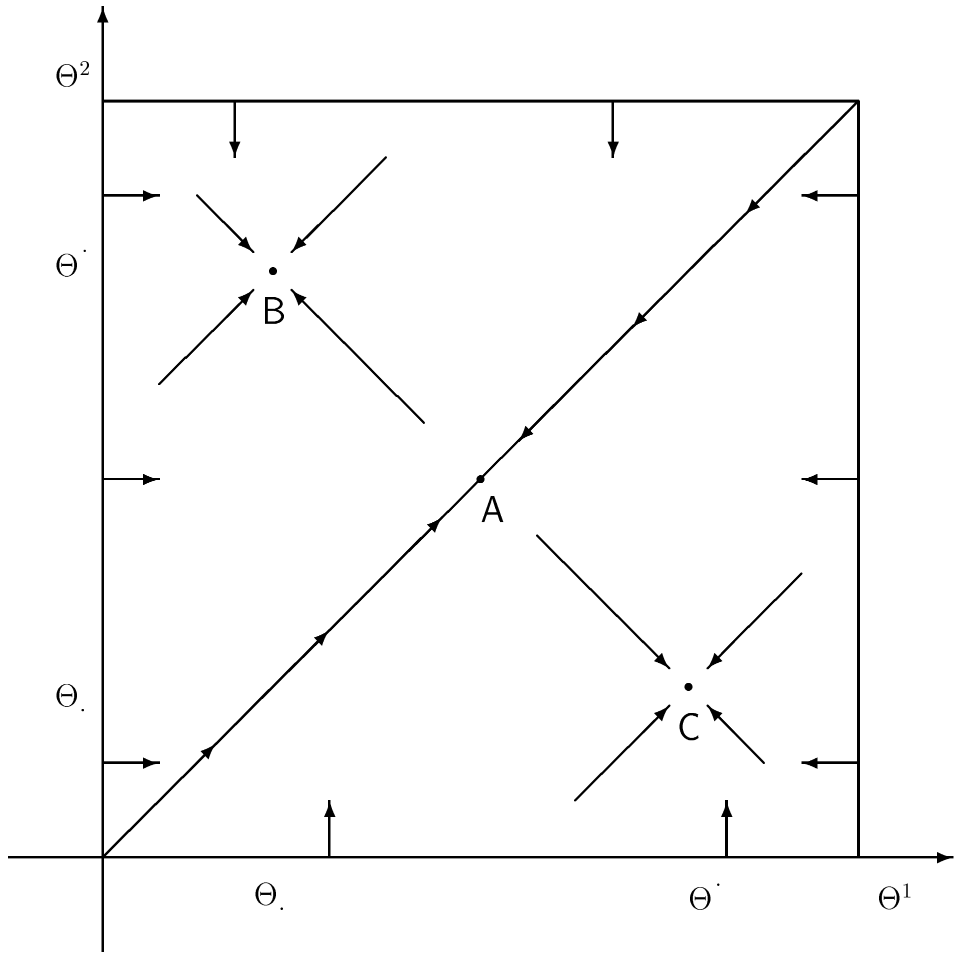

Figure 1: Phase-plan for the dynamical system of Eq.(3.8)

To begin with, let's show that the diagonal is an invariant set for the associated flow. Take $\theta^1(t_0) = \theta^2(t_0) = \theta_0$ as an initial condition. The symmetric solution to Eq. (3.7) must be of the form:

$$P^* = \frac{P^*_y}{P^*_x} = \frac{\alpha_x(\theta_0)}{\alpha_y(\theta_0)} \tag{3.9a}$$

$$M^1 = M^2 = P^*_x \alpha_x(\theta_0) = P^*_y \alpha_y(\theta_0) \tag{3.9b}$$

$$x^1 = \beta\alpha_x(\theta_0), \quad x^2 = \beta\alpha_x(\theta_0) \tag{3.9c}$$

$$y^1 = (1-\beta)\alpha_y(\theta_0), \quad y^2 = (1-\beta)\alpha_y(\theta_0) \tag{3.9d}$$

and, by substituting into Eq. (3.8), we conclude

$$\dot{\theta}^i\big(t \mid \theta^1(t_0) = \theta_0\big) = \dot{\theta}^2\big(t \mid \theta^2(t-0) = \theta_0\big)$$

for all t. Hence, $\theta^1(t) = \theta^2(t)$ forever. Moreover, there exists a unique, attracting stationary state at $(\bar{\theta}, \bar{\theta})$ where $\bar{\theta}\colon [f(\beta) + g(1-\beta)]/\gamma$. This is point A in Figure 1.

Now let's consider an asymmetric initial condition, say, $\theta^1(t_0) > \theta^2(t_0)$ and assume x is the industrial good. Our basic intuition on the different speeds of learning-by-doing requires:

Assumption i *For every $\ell \in (0,1]\colon f(\ell) > g(\ell)$.*

Assumption ii *The function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$, defined as $\alpha(\theta) = \alpha_x(\theta)/\alpha_y(\theta)$ is increasing.*

Assumption i guarantees that, when initial conditions are different, comparative advantage will be important. Linearity of the technology implies that, for $\theta^1(t_0) > \theta^2(t_0)$, either country 1 specializes in production of good x or country 2 specializes in production of good y or both. The analysis may become rather complicated if we seek a complete description of each single case. Given the illustrative purposes of this example, we consider the situation in which both countries fully specialize. Notice, anyhow, that the qualitative conclusions will not be affected in the general case. Competitive equilibrium quantities and prices at any time $t \geq t_0$ will therefore be:

$$P^* = \frac{P_y^*}{P_x^*} = \frac{1-\beta}{\beta}\,\frac{\alpha_x(\theta^1)}{\alpha_y(\theta^2)} \tag{3.10a}$$

$$M^1 = p_x^*\alpha_x(\theta^1); \qquad M^2 = p_y^*\alpha_y(\theta^2) \tag{3.10b}$$

$$x^1 = \alpha_x(\theta^1); \quad x^2 = 0 \tag{3.10c}$$

$$y^1 = 0; \quad y^2 = \alpha_y(\theta^2) \tag{3.10d}$$

and the two countries will grow according to:

$$\dot{\theta}^1 = f(1) - \gamma\theta^1 \tag{3.11a}$$

$$\dot{\theta}^2 = g(1) - \gamma\theta^2 \tag{3.11b}$$

Denote $\theta^* = f(1)/\gamma$ and $\theta_* = g(1)/\gamma$, then $\theta^* > \theta_*$ because of Assumption i and $\theta^* > \bar{\theta}$ if $f(1) > f(\beta) + g(1-\beta)$. It is also immediate to see that, in fact, the two new steady-state positions $B = (\theta^*, \theta_*)$ and $B' = (\theta_*, \theta^*)$ are the mirror image of each other and are locally attractive. The basin of attraction of B is given by all the points (θ^1, θ^2) with $\theta^1 > \theta^2$ and that of B' is the other half of the positive orthant.

Notice also that, because of the crude simplifications we have been using, two other stationary positions in fact exist at $(\bar{\theta}, 0)$ and $(0, \bar{\theta})$. When one of the two countries has no expertise at time t_0 (say, $\theta^1(t_0) > 0$ and $\theta^2(t_0) = 0$), then the other country reverses to autarky and moves to the steady-state level $\bar{\theta}$, whereas the poor country "disappears" from the scene.

Figure 1 contains the bulk of our argument: in the presence of externalities generated by learning-by-doing mechanism and with differential products, free trade and competitive behavior tend to magnify small differences in the initial conditions and may easily lead to huge disparities in the long run.

4 The Dynamics of the General Model

As we said at the end of Section 2, we need a little more structure to be able to consider the vector-field of Eq. (2.8). We will do it here in order to show that the conclusions we have reached in the previous section may indeed persist under a more general formulation. The analysis will not be, in any case, exhaustive nor will we try to be rigorous in all our assertions. A formal treatment of the problem in terms of "Proposition-proof" requires additional work.

From Section 3 we keep the preferences' specification. The technology is as described in (T.1)–(T.2). In order to simplify the analysis, we find it more attractive to change variables and consider the dynamic processes in terms of new variables α^i and ω^i that will soon be defined.

Let a path for $\theta^1(t)$, $\theta^2(t)$ be given. This will induce a path $\alpha_x\big(\theta^i(t)\big)$ and $\alpha_y\big(\theta^i(t)\big)$, for $i = 1, 2$. Set $\alpha^i(t) = \alpha_x\big(\theta^i(t)\big)$ and define $\alpha^i_y = h(\alpha^i)$. This is always possible as we are considering monotone functions. Also set $\omega^i = x^i/\alpha^i = F_x(z^i_x)$, i.e., ω^i is an "aggregate index" of the amount of resources country i invests in the production of good x. For each ω^i let

$$T(\omega^i) = \sup\{f_y(z^i_y),\ \text{s. t. } \omega^i = F_x(z^i_x),\ z^i_x + z^i_y = 1\}$$

i.e., T describes the "Production Possibility Frontier" (PPF) when $\alpha^i_x = \alpha^i_y = 1$, a level which may very well not correspond to any θ^i. We will directly assume T to be strictly concave and differentiable, but this could be derived from a slight strengthening of our assumptions in Section 2. For $i = 1, 2$ given α^i (or equivalently θ^i), we may now write $y^i = h(\alpha^i)T(\omega^i)$ with $h(v) = \alpha_y\big(\alpha_x^{-1}(v)\big)$. As we observed above, the "instantaneous" competitive equilibrium is Pareto efficient and, hence, we must have for interior solutions:

$$\frac{P_x}{P_y} = -\frac{dy^i}{dx^i}|_T = -\frac{h(\alpha^i)}{\alpha^i}T'(\omega^i) = p \tag{4.1}$$

for both countries. We will in fact, for simplicity, assume in this section that interiority prevails. This would follow from Assumptions 1 and 2 of Section 2 if we further impose the classical "Inada Conditions."

We need now to redefine our dynamical system. As we have chosen α^1, α^2 as the two-state variables, we write

$$\dot{\alpha}^i = \phi(\omega^i) - \gamma\alpha^i, \qquad i = 1, 2. \tag{4.2}$$

Note that each ω^i depends on both α^1 and α^2 so that the new dynamical system is not decoupled. Moreover we are not, implicitly, making expertise sector specific: ω^i also determines the amount of resources used in the y-producing sector so that the form of Eq. (4.2) amounts to nothing more than a renormalization. The idea that x is the advanced sector is then conveyed in the new framework by the assumptions:

Assumption 1 *ϕ is positive and increasing.*

Assumption 2 *$h(\alpha)/\alpha$ is decreasing.*

In order to avoid unbounded growth (not an undesirable feature, but not our concern here), we also impose:

Assumption 3 *ϕ is bounded and $h(\alpha)/\alpha$ is also bounded.*

Once again let's consider what happens when the initial conditions are on the diagonal. Clearly if $\alpha^1 = \alpha^2$, then also $\omega^1 = \omega^2$. Moreover, because of the homothetic nature of the utility function, it is possible to show that the level of the ω^i's are constant over time and equal to the (unique) solution to the fixed-point problem: $T(\omega)/\omega = -T'(\omega) = -T'(\omega)(1-\beta)/\beta$, (uniqueness

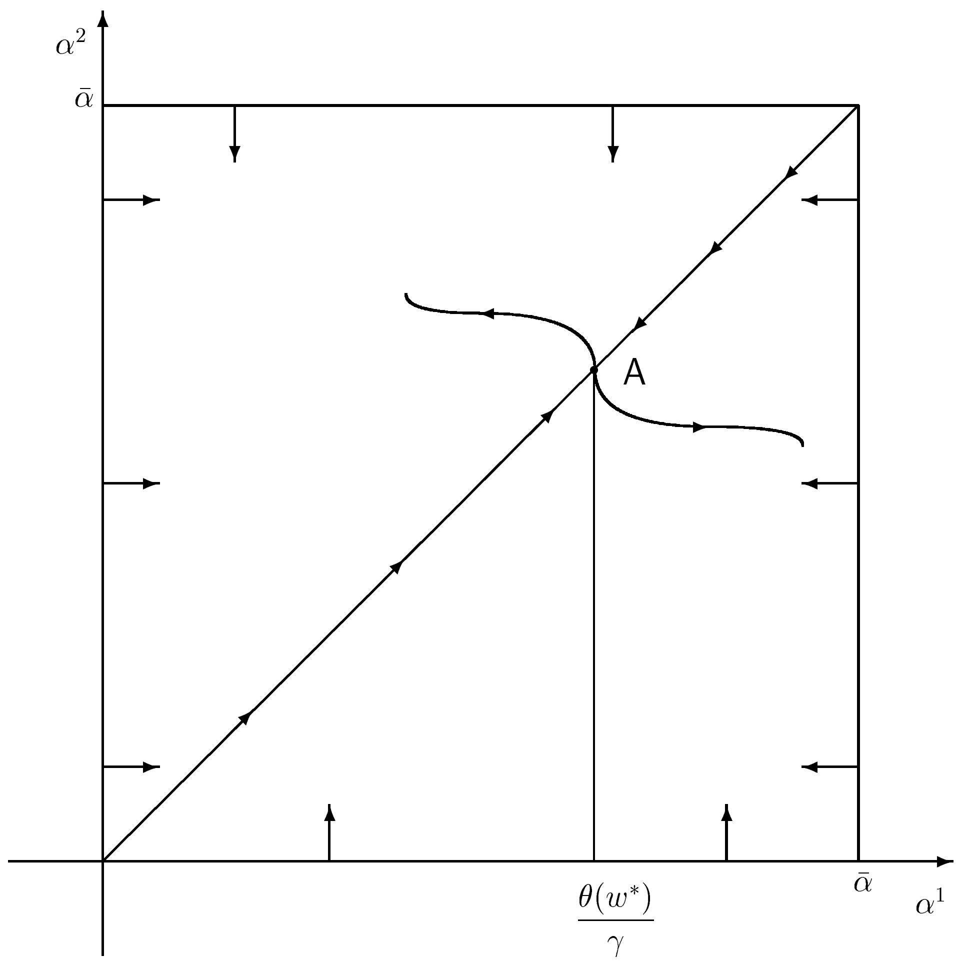

Figure 2: Phase-plan for the dynamical system of Eq.(4.2)

here follows from the concavity of T). Call this value ω^*; the dynamics on the diagonal is then

$$\dot{\alpha}^i = \phi(\omega^*) - \gamma\alpha^i, \qquad i = 1, 2, \tag{4.3}$$

so that a rest point will exist and all the orbits on the diagonal will converge to it. This is point A in Figure 2.

Before moving ahead and considering asymmetric initial conditions, let's pause and outline our strategy. We want once again to show that A is a saddle point for the vector field of Eq. (4.2) and that such a vector field points inward on the boundaries of some appropriate square $[0, \bar{\alpha}] \times [0, \bar{\alpha}]$ in the (α^1, α^2) plane. If this is the case, then standard Hopf-Poincaré degree arguments will guarantee that an odd number of equilibria (rest points) for Eq. (4.2) will exist.

Given the general nature of the functions ϕ, F_x, F_y, α_x and α_y, we have no method for computing the number of such equilibria and their dynamic stability. But this will not affect our qualitative argument: we may have other saddle points on each side of the diagonal, or a unique attracting cycle, or even sinks and sources and limit cycles (either stable or unstable); *in any case*, the competitive equilibrium paths will share the common feature of being asymmetric either because they converge to an asymmetric rest point or because they cycle along a closed curve that (being all on one side of the 45° line) will exhibit average levels of the α's (θ's) that are different across countries. And this qualitative behavior is exactly what we have in mind.

Set $\bar{\omega} = F_x(z^i)$ (remember that $z^1 = z^2$), then

$$\bar{\alpha} = \frac{\phi(\bar{\omega})}{\gamma} \tag{4.4}$$

is the maximum sustainable level of α for both countries and, clearly, the vector field of Eq. (4.2) points inward from any point of the type $(\bar{\alpha}, \alpha^2)$ and $(\alpha^1, \bar{\alpha})$ for $\alpha^i \in [0, \bar{\alpha}]$ (see Figure 2). Pointing inward from the other side requires further assumptions. In fact, we may have a situation in which a point of the type $\big(\phi(\omega^*)/\gamma, 0\big)$ (respectively $\big(0, \phi(\omega^*)/\gamma\big)$ will attract all the trajectories starting in an ϵ-neighborhood of the horizontal axis (respectively, vertical). Notice that such a feature is not necessarily harmful to our argument, as such a rest point is, indeed, an equilibrium where one country is much richer than the other. Nevertheless, we may like a model with less extreme predictions. It is not difficult to see what is required to guarantee our result. One sufficient condition is the technology to be such that you can produce something even when your expertise is zero and that you, in fact, choose to do so. This may be obtained either because some resources are always allocated to the production of good x, so that $\omega(\alpha^1, \alpha^2) > 0$ everywhere (i.e., total specialization never occurs) and/or because $\phi(0) > 0$, i.e., that even if all the resources are employed in the production of y, some (gross) expertise is acquired that can be used in sector x. Then, as long as $\alpha^i < \phi(0)/\gamma$, $\dot{\alpha}^i > 0$ is obtained. Another more subtle argument can be developed by using the "Inada" conditions mentioned above, together with the boundedness of $h(\alpha)/\alpha$, to ensure that no matter how small α^i is, provided $\alpha^j \leq \bar{\alpha}$ for $j \neq i$, $\omega(\alpha^i, \alpha^j) > \epsilon$. For then, if $\alpha^i < \phi(\epsilon)/\gamma$, $\dot{\alpha}^i > 0$.

We will simply assume the first case to be realized:

Assumption 4 *Either* $\phi(0) > 0$ *or* $\omega(\alpha^1, \alpha^2) > 0\, \forall(\alpha^1, \alpha^2 \in I\!R^2$ *and* $\phi(\omega) > 0$ *for* $\omega \in (0, \bar{\omega}]$.

This understood, we may proceed to the last step. Consider the case in which $\alpha^1(t_0) > \alpha^2(t_0)$, then as $h(\alpha)/\alpha$ is decreasing and T is concave, we have $\omega^1 < \omega^2$ at t_0. This is our basic comparative advantages intuition and it follows from Eq. (4.1). Therefore, $\alpha^1(t_0) > \alpha^2(t_0)$ implies that $\phi(\omega^1) > \phi(\omega^2)$ at that point. Next we observe that, under our hypothesis on h and T, the conditions for applying the implicit function theorem hold in a neighborhood of $\big(\phi(\omega^*)/\gamma, \phi(\omega^*)/\gamma\big) = (\alpha^*, \alpha^*)$. (They, in fact, hold everywhere on $(0, \bar{\alpha}] \times (0, \bar{\alpha}]$ which is why we can define the dynamical system of Eq. (4.2).) Then write $\omega^i = f_i(\alpha^1, \alpha^2)$, for $i = 1, 2$. Since $\omega^i = \omega^*$ on the diagonal, we have that:

$$\frac{\partial f_i(\alpha^*, \alpha^*)}{\partial \alpha^1} + \frac{\partial f_i(\alpha^*, \alpha^*)}{\partial \alpha^2} = 0, \qquad i = 1, 2. \tag{4.5}$$

From Eqs. (4.1) and (4.3) we have that $\alpha^1 > \alpha^2$ implies $\omega^1 > \omega^2$. This and Eq. (4.5) yields:

$$\frac{\partial f_i}{\partial \alpha^i} \geq 0, \quad \frac{\partial f_i}{\partial \alpha_j} \leq 0, \qquad i \neq j. \tag{4.6}$$

To exclude that the derivatives in Eq. (4.6) are both zero, we need to consider again the Competitive Equilibrium condition of Eq. (4.1). Write it as

$$F(\alpha^1, \alpha^2, \omega^1, \omega^2) = \frac{h(\alpha^1)}{\alpha^1} T'(\omega^1) - h\frac{h(\alpha^2)}{\alpha^2} T'(\omega^2) = 0 \tag{4.7}$$

and use the implicit function theorem to compute $\partial f_i / \partial \alpha^i = \partial \omega^i / \partial \alpha^i$. As we have assumed $h(\alpha)/\alpha$ to be decreasing, this is nonzero; therefore, both derivatives in Eq. (4.6) are nonzero in a neighborhood of (α^*, α^*) and equal in modulus.

To check under which conditions (α^*, α^*) is a saddle, we need only to linearize Eq. (4.2) aroud the symmetric equilibrium. The Jacobian computed, there is

$$\gamma\left\{\gamma - \phi'(\omega^*)\left[\frac{\partial f_1(\alpha^*, \alpha^*)}{\partial \alpha_1} + \frac{\partial f_2(\alpha^*, \alpha^*)}{\partial \alpha_2}\right]\right\} \tag{4.8}$$

As one root is certainly negative, we need Eq. (4.8) to be negative. Notice that the symmetry of the equilibrium can be used to simplify Eq. (4.8) so that our necessary and sufficient condition reads:

$$\gamma < 2\phi'(\omega^*)\frac{\partial f_1(\alpha^*, \alpha^*)}{\partial \alpha_1}. \tag{4.9}$$

It is easy to construct examples verifying Eq. (4.9) since one may, for instance, increase $\phi'(\omega^*)$ without altering either α^* or γ. If acquired expertise depreciates too fast with respect to its rate of self-reproduction, any divergent path is bound to snap back eventually. When the learning-by-doing mechanism is of some relevance (why bother otherwise?), then asymmetric equilibria are the logical outcome of our simple model. Notive, finally, that the positive synergies occurring from trade are reflected in Eq. (4.9) by the fact that the term on the right side sums up the effects from both countries. This suggests that in a general n-countries, m-commodities world, the asymmetric effects are more likely to dominate.

Finally, we show that in fact income of the most productive country will be the largest independent of tastes. Let M^i be the income of country i, i.e.,

$$M^i = p_x \alpha^i \omega^i + p_y h(\alpha^i) T(\omega^i).$$

From Eq. (4.1) we have that

$$\frac{M^1}{M^2} = \frac{h(\alpha^1)}{h(\alpha^2)} \frac{T(\omega^1) - T'(\omega^1)\omega^1}{T(\omega^2) - T'(\omega^2)\omega^2}.$$

If $\alpha^1 > \alpha^2$, then as observed above $\omega^1 > \omega^2$. Further, by the strict concavity of T:

$$T(\omega^1) - T'(\omega^1)\omega^1 > T(\omega^2) - T'(\omega^1)\omega^2 \geq T(\omega^2) - T'(\omega^2)\omega^2.$$

Since $h(\alpha^1) \geq h(\alpha^2)$, we have that $M^1 > M_2$.

5 Conclusions

We refrain from deriving too many implications from such a simple model and, in particular, to discuss the kind of government policies—production subsidies, import tariffs—that could ameliorate the dynamic inefficiencies. Better insights should come from a more articulated analysis that we are already developing. We only point out a few remarkable limits of this exercise, limits we hope to be able to overcome in the near future.

1. One may want more definite predictions. This will require the choice of "reasonable" functional forms and, very likely, the use of numerical simulations.

2. The shortcomings of the log utility functions outlined at the end of Section 2 must be eliminated by the choice of more sophisticated and more flexible specification of the utility function. The linearity of the Engel curve with respect to income is an especially disturbing limitation.

3. The notion of expertise/knowledge must be analyzed more deeply and cannot be relied upon as the only "engine" of growth. This will amount to an explicit consideration of the capital accumulation process which will yield, in turn, a real intertemporal optimizing framework. The notion of human capital and the ways in which "social knowledge" is embodied in "objects" and transferred over time is an important, related topic.

4. Finally, we may want to allow borrowing-lending to occur across the two countries. This will enable intertemporal consumption smoothing and may therefore affect the temporal pattern of demand and prices. The dynamical system to be considered in this case is a three-dimensional one and is not *a priori* clear that the same simple conclusions will replicate.

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