COMPETITIVE EQUILIBRIUM GROWTH

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ABSTRACT. We construct an abstract, dynamic general equilibrium model of innovation, growth and cycles in the spirit of Schumpeter's *Theory of Economic Development*. Despite the existence of infinitely many commodities and activities, the use of which may increase over time, we give a standard characterization of equilibrium using the first and second welfare theorems, and a standard transversality condition. We consider a series of examples characterizing the dynamic properties of equilibria and show that many results discussed in the "endogenous growth" literature can be obtained as special cases of the model we propose.

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1. INTRODUCTION

We construct an abstract, dynamic general equilibrium model of innovation, growth and cycles in the spirit of Schumpeter's *Theory of Economic Development*. Despite the existence of infinitely many commodities and activities, the use of which may increase over time, we give a standard characterization of equilibrium using the first and second welfare theorems, and a standard transversality condition. We consider a series of examples characterizing the dynamic properties of equilibria and illustrate which results discussed in the development literature may or may not be obtained as special cases of the model we propose.

We study an economic environment with an activity analysis technology in which the number of (produceable) commodities is a countable infinity. Similarly, there exist a countable infinity of potential "activities". An activity is characterized by a pair of input and output vectors and displays constant returns to scale. The input goods used in production come from output of the previous period. The level at which an activity is operated is limited by the availability of inputs and by aggregate demand and, therefore, relative prices.

We are not interested in modelling the way in which activities, in the sense of "blueprints" or "ideas", have come about. While this generative process has clear economic content and probably obeys principles not so different from those characterizing the production and trade of various species of potatoes or derivative securities, our current understanding of its operational rules is too primitive to make the theoretical modelling anything more than trivial. Of course we can pretend that one or more of our abstract "activities" is concerned with the production of scientific inventions or with R&D. We prefer to take the set of available ideas as exogenous to the economic process, which is instead concerned with their (profitable) implementation into production processes.

There exists a finite number of types of infinitely lived consumers. They derive utility from being able to enjoy a (possibly ever-increasing) amount of "characteristics", as in Lancaster [1966]. There exists a finite number of such characteristics and each commodity is identified with a vector of them. So while the number of commodities is infinite, the characteristics they produce are finite, and our utility functions are standard additively separable utility functions, with period utility a function on a fixed finite dimensional space.

At each point in time only a finite number of goods is being produced through a finite number of activities. Barring the implementation of new ones, this defines a neoclassical infinite horizon production economy in the spirit of McKenzie [1981, 1986]. For any such "McKenzie Economy", well known assumptions about discounting, concavity of the payoff and convexity of the feasible set should enable one to prove a Turnpike Theorem: equilibrium trajectories converge asymptotically to a balanced-growth path, (Bewley [1982] and Yano [1984]).

In the full-blown economy, though, convergence to any balanced growth path is, at best, partial and temporary as new feasible activities are implemented when they become profitable and other are disbanded when not profitable anymore. Therefore, economic growth occurs through a sequence of (endogenous) cycles, and technological "shocks" are the results of the never ending process of entrepreneurial innovation.

An entrepreneurial innovation always consists in the activation of a new activity and is carried out by individual agents. In practice this may result in the introduction of a new good or, simply, in the production of an old one by means of a different combination of inputs. The vector of available capital stocks restricts the set of available activities at any given point in time.

An initial condition consists in a vector of capital stocks and in their allocation among the different types of households. An intertemporal equilibrium is defined as a feasible production/consumption sequence along with prices which satisfy utility maximization, market clearing and the no-profit condition.

The model economy so constructed exhibits a number of interesting features.

(a) Economic growth is endogenous but not predetermined: initial conditions affect the development path. Also, the model does not have a unique predetermined order in which new goods have to be introduced. The notion of "path dependence" becomes therefore relevant in such context.

(b) Unlimited economic growth does not translate into the continuous accumulation of an infinitely large quantity of the same good(s). Instead, it consists in the increasing satisfaction of a limited number of desires or necessities (the characteristics) by means of an ever changing set of goods. Stokey [1988] has already studied a one-dimensional version of this story. In our multi-commodities world, new goods can be priced "by arbitrage" using baskets of other goods delivering the same set of charactericts. Under such a pricing scheme, "profits" are the outcome of succesfull entrepreneurial efforts and appear as rents to certain factors of production.

(c) When interpreted in this manner, profits become consistent with competitive equilibrium and Pareto efficiency and become the engine of economic growth. The implementation of new goods and activities is explained by the desire of increasing the size of the rents accruing to one's factors of production and reducing those of the others. This depends, in the end, on the type of allocational mechanisms that are in place, in our case perfectly competitive markets. This should make the model suitable for studying which economic institutions and systems of property rights favor or hamper the process of economic development.

(d) The same mechanisms that generate development may prevent it, even when technologically feasible. The importance of market arrangements, trade-openess, income distribution, pre-existence of certain capital goods, availability of credits, etc. are all issues one can address within specialized versions of the model.

(e) Economic growth and business cycles turn out to be the joint outcome of the same set of economic and technological changes. In this sense they are "endogenous" and determined by the actions of economic agents-entrepreneurs seeking to generate the maximum amount of profits.

The organization of the rest of the paper is the following. The next Section introduces the notation and lists the main assumptions of the abstract model. Section 3 characterizes the competitive equilibra as solutions to a social optimum and shows that standard versions of the First and Second Welfare Theorems do apply. In Section 4 we develop and discuss some implications of the zero-profit condition for the pattern of economic innovation and the long-run growth rate of output, and discuss the way in which new commodities are priced and fixed factors' rents emerge and dissipate. In Section 5 we check, by means of examples, which of the main results of the "endogenous growth" literature can be replicated in our setup and which one cannot; while Section 6 begins to use special cases of the general model to address some classical discussions on the sources of economic growth. Finally Section 7 does the same with respect to business cycle theory. The last Section concludes.

2. THE ABSTRACT MODEL

2.1. **Households.** We consider an infinite horizon economy, $t = 0, 1, 2, \ldots$ with a finite number of different types of consumers $h = 1, \ldots, H$. There is a continuum of size one for each type. Consumer h values characteristics $c_t^h \in \Re_+^J$ where J is the number of characteristics. The *period utility* received in a particular period from the consumption of characteristics is denoted $u^h(c_t^h)$.

Assumption 2.1. The period utility $u^h(\cdot)$is strictly increasing, concave, smooth and bounded below. Total *lifetime utility* is given by $U^h(c^h) = \sum_{t=1}^{T} \delta^{t-1} u^h(c_t^h)$, where $0 \leq \delta < 1$ is the common subjective discount factor.

The assumptions that the utility function is strictly increasing and concave are standard. The smoothness of the period utility function is convenient and, for a concave function, not terribly restrictive. The assumption that the period utility function is bounded below is technically useful. It insures that U is well defined (although possibly infinite). As we are concerned with the theory of growth, not the theory of subsistence, we are primarily interested in the behavior of $u^h(\cdot)$ for large and possibly growing quantities of consumption, so the behavior of the utility function near $c^h = 0$ is rather secondary to our ends. Moreover, from an intellectual perspective, if $u^h(0) = -\infty$ this has the counterfactual implication that no amount of consumption, however large, will compensate for any probability of $c^h = 0$ no matter how small this probability might be. It is true that the assumption of boundedness below rules out some CES utility functions such as the logarithm. However, for any utility function $\tilde{u}^h$ that is strictly increasing, concave and smooth and any $\tilde{c}^h > 0$ we may introduce a new utility function $u^h_{\tilde{c}^h}$ that is increasing, concave, smooth and bounded below and such that $\tilde{u}^h(c^h) = u^h_{\tilde{c}^h}(c^h)$ for all $c^h \geq \tilde{c}^h > 0$. In this way we can incorporate utility functions that behave, for example, like the logarithm for all except very low levels of consumption. When we use CES period utility functions in examples below, we will not explicitly introduce this construction, although it should be understood to be necessary in those cases in which utility would otherwise be unbounded below.

Characteristics c_t are acquired through the consumption of commodities. The potential number of commodities is countably infinite although, at each point in time, only a finite number of them will be produced or consumed. The *period commodity space* consists of the subset of $X \subseteq \ell^+_\infty$ of sequences $(x_1, x_2, \ldots, x_n, \ldots) \geq 0$ for which $x_n = 0$ for all but finitely many n. The overall *commodity space* is then

$$\tilde{X} = \times_{t=0}^{\infty} X$$

The vector of characteristics acquired by consumption of a single unit of commodity n is denoted by $C_n \in \Re^J_+$. These induce a linear map $C : X \to \Re^J_+$. So, if x_t denotes the vector of commodities consumed at time t the characteristics enjoyed by the agent are $c_t = Cx_t$. We also denote by C^i the map $C^i : X \to \Re_+$ from commodity vectors to the amount of characteristic i acquired.

2.2. **Production.** Production takes place through linear activities. An activity a is a couple of vectors $(k(a); y(a))$ where $k(a) \in X_+$ denotes the input of commodities entering the activity at the end of period t, and $y(a) \in X_+$ the output of commodities made available by the activity at the beginning of the following period. During period $t+1$ the outputs can either be consumed or used as inputs for further production.

The whole set of potential activities is countable, and is denoted by A. At each point t in time there exists a set $A_t \subseteq A$ of activities which are available at that time. The sequence $\{A_t\}_{t=0}^{\infty}$ is taken as given.

Assumption 2.2. $A_0 \neq \emptyset$; for all t $A_t \subseteq A_{t+1}$and A_t is finite.

Activities can be simultaneously operated at every non negative level $\lambda(a)$ as long as inputs $\lambda(a)k(a)$ are available. We assume that A satisfies *no-free-lunch* and it allows for *free disposal*[1]

Assumption 2.3. For all $a \in A$ if $k(a) = 0$, then $y(a) = 0$; for all $\varepsilon \in [0,1]$ and $\theta \geq 1$if $(k(a); y(a))$ is an activity then also $(\theta k(a); \varepsilon y(a))$ is an activity

Denote the vector of activity levels at time t as $\lambda_t \in \Re^A_+$ and define the aggregate stock of capital at time t as $k_t = \sum_{a \in A_{t-1}} \lambda_{t-1}(a)y(a) - \sum_{h=1}^H x_t^h$.

Definition 2.4. The production plan ??? is a *feasible production plan* for the initial condition $k_0$if $\sum_{a \in A_t} \lambda_t(a)y(a) \geq k_{t+1} + \sum_{h=1}^{Hxh_{t+1}} k_t \geq \sum_{a \in A_t} \lambda_t(a)k(a)$ for all $t = 0, 0, \ldots$.

We call an activity $a \in A_t$ *viable* at t for initial condition k_0, if there exists a socially feasible production plan starting from k_0 and such that $\lambda_t(a) > 0$. We denote the set of viable activities at time t from k_0 with $A_t(k_0)$. Note that $A_t(k_0) \subseteq A_t$, but may in fact be a proper subset if, for some $a \in A_t$, we have $k_n(a) > 0$ and there is no feasible plan from k_0 such that the vector of outputs y_{t-1} has $y_{n,t-1} > 0$, where n is the index of a commodity. Similarly, we call a commodity n *viable* at time t for the initial condition k_0 if there exists a socially feasible production plan starting from k_0 and such that $k_{n,t} + x_{n,t}^h >$ 0.

2.3. **Optimality and Supporting Prices.**

Definition 2.5. Given $\mu = (\mu^1, \ldots, \mu^H) \geq 0$, the allocation $a^* = (\lambda^* \in (\times_{t=1}^\infty \Re^A_+)$, $k^* \in \tilde{X}_+$, $x^* \in \tilde{X}_+^H)$ **solves the social planner problem** for initial condition k_0 if it solves

$$\max_{\lambda,k,x} \ \mathcal{U}_\mu(c) = \sum_{h=1}^H \mu^h U^h(c^h)$$

subject to: $c_t^h = Cx_t^h$, and feasibility of the production plan.

Let $p_t \in \Re^\infty_+$ be the price of commodities at time t and $p \in \mathcal{P} = \times_{t=0}^\infty \Re^\infty_+$ a whole sequence of prices from time zero to infinity. In a competitive equilibrium with transfer payments, these prices must satisfy two conditions: they should yield zero profits and support the preferences.

[1]Notice that these disposal activities will never be used as there are no "bads" in the economy; and we largely ignore their existence in the sequel. However, they are a convenient way of guaranteeing non-negative prices, which we make heavy use of.

Definition 2.6. Given $k \in \tilde{X}_+$ and $\lambda \in \times_{t=0}^{\infty} \Re_+^A$ the prices $p \in \mathcal{P}$ satisfy the **zero profit condition** if,

$$p_{t+1} y(a) - p_t k(a) \leq 0, \forall a \in A_t(k_0),\ t = 0, 1, \ldots$$

with equality if $\lambda_t(a) > 0$. For future purposes this can be written as

$$\pi_{t+1}(a) = \lambda_t(a)[p_{t+1} y(a) - p_t k(a)] = 0$$

Given prices $p \in \mathcal{P}$ the sequence $x^{h*} \in \tilde{X}$ **solves consumer** h**'s maximization problem** if x^{h*} is the argmax of

$$\max U^h(c^h),$$

$$\text{subject} \quad \text{to}: \quad c_t^h = C x_t^h,\ \sum_{t=1}^{\infty} p_t x_t^h \leq \sum_{t=1}^{\infty} p_t x_t^{h*}$$

The pair $p \in \mathcal{P}$ and $x^{h*} \in \tilde{X}$ satisfy the **first order conditions** for consumer h if there exists a $\mu^h \in \Re_+$ such that

$$p_{nt}^* \geq \mu^h \delta^{t-1} D u^h(C x_t^{h*}) C_n$$

with equality unless $x_{nt}^{h*} = 0$.

The pair $p \in \mathcal{P}$ and $k^* \in X$ satisfy the **transversality condition** if

$$\lim_{T \mapsto \infty} p_T k_T^* = 0$$

The feasible production plan ??? and the price sequence $p \in \mathcal{P}$ are a **competitive equilibrium with transfer payments** if they satisfy the zero profits condition and solve consumer h's maximization problem for all $h = 1, \ldots, H$.

3. Theory

We begin with a statement of the First and Second Welfare Theorems which fit our purposes.

Theorem 3.1. *Suppose assumptions 1, 2 and 3 hold. Suppose that λ^*, k^*, x^* is a feasible production plan given k_0 and that $\sum_{t=1}^{\infty} \delta^{t-1} u^h(C x_t^h) < \infty$. Then the following four conditions are equivalent:*

(1) λ^*, k^*, x^* are Pareto efficient for initial condition k_0.

(2) λ^*, k^*, x^* solve the planner's problem for initial condition k_0 and some non-negative weights μ.

(3) There exist prices p^* satisfying the zero profit condition and such that x^{h*} solves the consumer maximization problem given p^* with $\sum_{t=1}^{\infty} p_t^* x_t^{h*} < \infty$ for all h.

(4) There exist prices p^* satisfying the zero profit condition such that the pair p^* and x^{h*} satisfies the first order conditions and the pair p^* and k^* satisfies the transversality condition.

Proof. First we observe that if the zero profit condition holds then the transversality condition is true if and only if $\sum_{t=1}^{\infty} p_t x_t^h \leq \sum_{t=1}^{\infty} p_t x_t^{h*}$. Indeed from the zero profit condition

$$p_0^* k_0^* - p_{T+1}^* k_{T+1}^* = \sum_{t=0}^{T} (p_t^* k_t^* - p_{t+1}^* k_{t+1}^*) = \sum_{t=0}^{T} p_{t+1}^* \sum_{h=1}^{H} x_{t+1}^{h*}$$

Second, we observe that (1) if and only if (2) is immediate from the fact that the feasible set of individual utility vectors is convex, compact and has non-empty interior.

To show that (4) implies (3) define the T truncated utility function by $U^{hT}(c^h) = \sum_{t=1}^{T} \delta^{t-1} u^h(c_t^h)$. Under Assumption 3, the truncated first order conditions

$$p_{nt}^* \geq \mu^h \delta^{t-1} Du^h(Cx_t^{h*})C_n \quad \text{with equality unless} \quad x_{nt}^{h*} = 0$$

are necessary and sufficient for the maximization of $\mu^h U^{hT}(c^h) + p_{T+1}^* k_{T+1}^*$ subject to

$$c_t^h = Cx_t^h, \quad \sum_{t=1}^{T} p_t^* x_t^h + p_{T+1}^* k_{T+1} \leq \sum_{t=1}^{T} p_t^* x_t^{h*} + p_{T+1}^* k_{T+1}^*$$

Since $\sum_{t=1}^{\infty} p_t x_t^h \leq \sum_{t=1}^{\infty} p_t x_t^{h*}$ if and only if the transversality condition is satisfied, we see that if x^{h*} does not solve the consumer maximization problem for some agent h, then there is a budget feasible $\hat{x}^h$ that yields more utility, i.e. $U^h(\hat{x}^h) > U^h(x^{h*})$. The pair $\{\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_T\}$ and $k_{T+1} = 0$ is budget feasible also in the truncated consumer problem, so that along a subsequence

$$\mu^h U^{hT}(Cx^{h*}) + p_{T+1}^* k_{T+1}^* \geq U^{hT}(C\hat{x}^h)$$

Since $U^{hT}(Cx^{h*}) \to U^h(Cx^{h*})$ and $p_T^* k_T^* \to 0$ we find in the limit that $U^h(Cx^{h*}) \geq \limsup_{T \mapsto \infty} U^{hT}(C\hat{x}^h)$, the desired contradiction, establishing (4) implies (3).

A standard proof of the first welfare theorem shows that (3) implies (2).

To show that (2) implies (4), suppose that λ^*, k^*, x^* is in fact a solution to the planner problem for the utility weights $(\mu^1, \ldots, \mu^H)$ and initial condition k_0. Observe that (when appropriately truncated) λ^*, k^*, x^* solves the problems of maximizing $\sum_h \mu^h U^{hT}(Cx_t^h)$ subject to social feasibility and $k_{T+1} \geq k_{T+1}^*$. Again by assumption 3 this is a finite dimensional problem for which we can find prices p^T so that the first order and zero profit conditions are satisfied. Moreover, the largest component of p_0^T corresponding to a non-zero element of k_0 is bounded above and below by some B_0 and $1/B_0$ respectively. From the zero profit condition, it follows that for each t there is a number B_t such that the largest component of p_t^T corresponding to a commodity viable at time t is less than or equal to $\prod_{i=0}^{t} B_i$. It follows, by passing to a subsequence if necessary, that there is a limit p^* as $T \to \infty$ such that the zero profit and first order conditions are satisfied.

To check the transversality condition recall it is equivalent to $\sum_{t=1}^{\infty} p_t x_t^h \leq \sum_{t=1}^{\infty} p_t x_t^{h*}$which, in turn, is equivalent to

$$\sum_{t=1}^{\infty} \delta^{t-1} Du^h(Cx_t^{h*})Cx_t^{h*} < \infty$$

Since u^h is concave and bounded below by $u^h(0)$ we have that $u^h(Cx_t^{h*}) \geq u^h(0) + Du^h(Cx_t^{h*})Cx_t^{h*}$, and so

$$\sum_{t=1}^{\infty} \delta^{t-1} Du^h(Cx_t^{h*})Cx_t^{h*} \leq \sum_{t=1}^{\infty} \delta^{t-1}[u^h(Cx_t^{h*}) - u^h(0)] < \infty$$

□

The nature of both the social optimum and the supporting prices can be illustrated by a simple example with vintage capital.

3.1. **Example - The Vintage Capital Model:** In this example there is a single characteristic and a single consumer so that $J = H = 1$. The period utility function is $u(c) = -(1/\theta)[c-1]^{-\theta}$. There are two types of commodities, a single consumption good and an infinite sequence of different vintages of capital, indexed by $i = 0, 1, \ldots$. The consumption good may be converted into the desired characteristic on a 1–1 basis. We write a commodity vector $x = (z, \kappa)$ where z is a scalar denoting the consumption good and κ is an infinite vector of capital stocks of different vintages, and we let the symbol χ_i denote the vector consisting of one unit of capital i and zero units of all other capitals. So, for example, $(0, \chi_2)$ is a commodity vector with 0 units of consumption, 1 unit of vintage 2 capital and zero units of everything else.

There are 2 sequences of activities. One sequence of activities, $[0, \chi_i; \gamma^i, 0]$, $\gamma \in \Re_+$ produces consumption from vintage i capital. The second, $[0, \chi_i; 0, \rho\chi_{i+1}]$, $\rho > 0$, produces ρ units of vintage $i+1$ capital from 1 unit of vintage i capital. The endowment is a single unit of vintage 0 capital.

Notice that in this setup, at time t there is only capital of vintage t, which is why we call this a vintage capital model. We look for a social optimum where capital (when measured in some appropriate physical unit) grows exponentially from one vintage to the next, and a constant fraction ϕ of the stock of current vintage capital is used in the production of the consumption good. Let κ_{tt} denote the amount of vintage t capital at time t. One unit of this capital can be used to produce γ^t units of consumption good, yielding a marginal present value utility of $\delta^t \gamma^t (\phi \gamma^t \kappa_{tt})^{-\theta-1}$. Alternatively it can be used to produce ρ units of vintage $t+1$ capital for time $t+1$, leading to a present value marginal utility of $\rho\delta^{t+1}\gamma^{t+1}(\phi\gamma^{t+1}\kappa_{t+1,t+1})^{-\theta-1}$, while $\kappa_{t+1,t+1} = \rho(1-\phi)\kappa_{tt}$. Equating payoffs from the two alternative uses of capital we get

$$1 = \delta(\gamma\rho)^{-\theta}(1-\phi)^{-\theta-1}$$

or

$$\rho(1-\phi) = (\delta\rho\gamma^{-\theta})^{1/(1+\theta)}$$

Notice that $\rho(1-\phi)$ is the growth rate of the capital stock; the growth rate of consumption is

$$\gamma\rho(1-\phi) = (\delta\rho\gamma)^{1/(1+\theta)}$$

Therefore, if $\delta(\delta\rho\gamma)^{-\theta/(1+\theta)} \geq 1$ this yields an infinite present value of utility; otherwise total lifetime utility is finite, which is what we require.

From $p^*_{nt} \geq \mu^h\delta^{t-1}Du^h(Cx^{h*}_t)C_n$, this gives price p^*_{zt} of the consumption commodity proportional to

$$p^*_{zt} \propto (\gamma\rho)^{-t}$$

Denote with $p^*_{\kappa_{tt}}$ the price of vintage t capital at time t. The zero profit condition for the consumption production activity is

$$p^*_{z,t+1}\gamma^t - p^*_{\kappa_{tt}} = 0, \qquad \text{or} \quad p^*_{\kappa_{tt}} = \frac{\rho^{-t-1}}{\gamma}$$

The profit from the activity of producing new capital is proportional to

$$\rho p^*_{\kappa_{t+1,t+1}} - p^*_{\kappa_{tt}} = \rho(\rho)^{-t-1} - (\rho)^{-t} = 0.$$

Finally, we check the transversality condition

$$[(\rho\gamma)^{-\theta}\delta]^{t/(1+\theta)} \to 0,$$

which is satisfied if

$$\delta < [\delta\rho\gamma]^{\theta/(1+\theta)}$$

The latter is equivalent to $\phi > 0$ and total lifetime utility to be finite, which makes sense. Summing up

Proposition 3.2. *Consumption of the characteristic grows at the rate* $g_c = (\delta\rho\gamma)^{1/(1+\theta)}$ *and the quantity of capital stock, from one vintage to the next, grows at the rate* $g_\kappa = (\delta\rho\gamma^{-\theta})^{1/(1+\theta)}$ *provided that* $\delta g_c^{-\theta} < 1$.

Notice that in the case in which utility is unbounded below (i.e $\theta \geq 1$) we are interested only in the case in which growth is non-negative, that is $g_c \geq 1$.

As long as $\theta > -1$ and $\gamma > 1$, the growth rate of the characteristics is higher than the growth rate of the physical stock of capital. In fact one may easily verify that there exists parameter values at which the characteristic grows and the physical amount of capital does not grow and may even shrink. This is a reasonable outcome: as long as new vintages of capital are more productive than old ones, ever growing consumption levels may be obtainable without filling the whole planet with production plants.

Notice that while this example is based upon the most primitive form of technological change, in which technology improves at a fixed exogenous rate, the actual growth rate of the economy is endogenous, since it depends on the rate of capital accumulation. In particular, if the economy becomes more productive (as measured by ρ or γ), or more patient (as measured by δ or θ^{-1}), it grows faster. The complaint that it has been made against this description of technological change is that it is insufficiently endogenous (innovations are implemented no matter what), and that there are no fixed factors as all commodities can be expanded indefinetely. We consider these limitations in the examples of Section 5.

4. Some Implications of Zero-Profits

One of our major goals is to understand how economic innovation takes place, i.e. how new activities and/or new commodities are introduced, and how this is connected with capital accumulation and the behavior over time of aggregate national output. It turns out that the zero profit condition has some important and straightforward implications for the introduction of new technologies, the pricing of new goods and the growth rate of National Product.

Given equilibrium prices p^* and an arbitrary production plan λ, we can measure the value at time 0 of total output available at time t as $\sum_{a \in A_t} \lambda_t(a) p^*_{t+1} y(a)$. The latter corresponds to the market value of Gross National Product (GNP) plus the stock of productive capital left over after depreciation. Net National Product (NNP) can also be computed as $p^*_{t+1}[\sum_{a \in A_t} \lambda_t(a) y(a) - k_t]$.

Economic development is often defined as a process of sustained growth in either GNP or NNP. One is therefore interested in the predictions of our model about the extent to which, along a competitive equilibrium path, new activities are made viable and then adopted and the impact this has on the long-run behavior of aggregate statistics such as GNP, NNP or total consumption.

We begin our analysis by looking at the choice of activities at a given point in time. Consider the situation at the end of period t, when the vector k_t of stocks of capital to be used in production is given and so is the set of viable activities A_t. Assume, to spare notation, that $\forall a \in A_t \ \exists \varepsilon > 0, \varepsilon k(a) \leq k_t$, so that every activity can be implemented at some positive level (if not, restrict attention to the subset for which this holds). Which, among the viable activities, will be used with a positive intensity?

Denote with Λ_t the collection of all production plans $(\lambda_t(a_1),\ldots,\lambda_t(a_N))$, $N = \sharp(A_t)$, starting at time t and which satisfy

$$\sum_{a\in A_t} \lambda_t(a)k(a) = k_t \tag{4.1}$$

Theorem 4.1. *Let λ^*, k^* and x^* be a solution to the social planner problem which is decentralized by the prices p^*. Let Λ_t be the set of time-t production plans satisfying (4.1) for $k_t = k_t^*$. Then, for all $(\lambda_t(a_1),\ldots,\lambda_t(a_N)) \in \Lambda_t$*

$$p^*_{t+1}\sum_{a\in A_t}\lambda^*_t(a)y(a) \geq p^*_{t+1}\sum_{a\in A_t}\lambda_t(a)y(a)$$

Proof. The zero profit condition implies that, for any λ_t other than λ_t^*

$$\sum_{a\in A_t}\lambda_t(a)[p^*_{t+1}y(a) - p^*_t k(a)] \leq 0$$

and that

$$\sum_{a\in A_t}\lambda^*_t(a)[p^*_{t+1}y(a) - p^*_t k(a)] = 0$$

The conclusion follows. □

In words: given a stock of productive inputs, the competitive equilibrium plan selects production activities and intensity levels in such a way that both total output and NNP are maximized. No "valuable" activity which is viable given the current stock of capital will be left unexploited.

The same simple logic leads to a much stronger result: the competitive equilibrium plan not only maximizes total output at a certain point in time and for a given stock of inputs, but also achieves the highest growth rate of total output in the very long-run.

To show this we begin by comparing the aggregate growth rate that obtains along a competitive equilibrium path with that of any other production plan that uses, at most, the same technologies as those used in the competitive equilibrium.

Theorem 4.2. *Let λ^*, k^* and x^* be a solution to the social planner problem which is decentralized by the prices p^*. Assume there exists a socially feasible plan $\hat{\lambda}$ such that*

$$(i) \qquad \lambda^*_t(a) = 0 \;\;\Rightarrow\;\; \hat{\lambda}_t(a) = 0;$$

$$(ii) \quad \sum_{a\in A_t}\hat{\lambda}_t(a)k_n(a) = \sum_{a\in A_{t-1}}\hat{\lambda}_{t-1}(a)y_n(a)$$

for all commodities n for which $p^*_{nt} \neq 0$. Then, for any other socially feasible plan λ

$$\sum_{a\in A_t} p^*_{t+1}\lambda_t(a)y(a) \leq \sum_{a\in A_t} p^*_{t+1}\hat{\lambda}_t(a)y(a)$$

for all $t = 0, 1, \ldots$.

Proof. Any socially feasible plan other than the solution to the social planner problem will yield non-positive profits at the supporting prices p^*, i.e.

$$\sum_{a \in A_t} \lambda_t(a)[p^*_{t+1} y(a) - p^*_t k(a)] \leq 0$$

Together with feasibility this implies that

$$p^*_{t+1} \sum_{a \in A_t} \lambda_t(a) y(a) \leq p^*_t \sum_{a \in A_{t-1}} \lambda_{t-1}(a) y(a), \quad t = 1, 2, \ldots$$

By construction the comparison path $\hat{\lambda}$ yields zero aggregate profits at the supporting prices, i.e.

$$\sum_{a \in A_t} \hat{\lambda}_t(a)[p^*_{t+1} y(a) - p^*_t k(a)] = 0$$

The latter and condition (ii) of the theorem imply that

$$p^*_{t+1} \sum_{a \in A_t} \hat{\lambda}_t(a) y(a) = p^*_t \sum_{a \in A_{t-1}} \hat{\lambda}_{t-1}(a) y(a), \quad t = 1, 2, \ldots$$

From which

$$p^*_{t+1} \sum_{a \in A_t} \lambda_t(a) y(a) \leq p^*_{t+1} \sum_{a \in A_t} \hat{\lambda}_t(a) y(a), \quad t = 0, 1, 2, \ldots$$

as both λ and $\hat{\lambda}$ are restricted by the same initial condition k_0. □

This theorem says that, given the set of activites used along the competitive equilibrium path, if the output of all the goods that have positive value can be entirely used as a production input, then one achieves the greatest socially feasible growth rate of total output and NNP in every single period.

This theorem is not entirely satisfactory because its second hypothesis badly fits the case in which there is joint production. As we show in an example at the end of the section, when there is joint production it may be impossible to divert all valuable output to saving and capital, and the condition of the theorem cannot be satisfied. Second, the theorem characterizes only the potential growth rate of NNP or GNP, and not the actual growth rate along an equilibrium plan.

Naturally, the actual growth rate is less than the potential growth rate, as some output of capital must be used for the production of consumption goods. In our vintage capital example, the capital stock grows at $\rho(1 - \phi)$, although potentially it could grow as fast as ρ if there were no consumption. Our next theorem fixes the share of total output used as investment, and shows that subject to this constraint, output grows at the fastest possible rate in competitive equilibrium.

Theorem 4.3. *If a solution to the social planner problem λ^*, k^*, x^* is decentralized by prices p^* and a socially feasible plan λ, k, x satisfies*

$$\frac{p_t^* k_t}{p_t^*(k_t + x_t)} \le \frac{p_t^* k_t^*}{p_t^*(k_t^* + x_t^*)}$$

then

$$\sum_{a \in A_t} p_{t+1}^* \lambda_t(a) y(a) \le \sum_{a \in A_t} p_{t+1}^* \lambda_t^*(a) y(a).$$

Proof. From the zero profit condition

$$\sum_{a \in A_t} \lambda_t(a)[p_{t+1}^* y(a) - p_t^* k(a)] \le 0$$

$$\sum_{a \in A_t} \lambda_t^*(a)[p_{t+1}^* y(a) - p_t^* k(a)] = 0$$

from which the result follows directly. □

This theorem shows that higher total output (and, as a corollary, higher NNP) than that from the competitive equilibrium plan is possible only by accepting a higher investment rate. In particular, the last theorem shows the following circumstance cannot arise. There is an activity that is expensive to operate in early periods (when it would earn a negative profit), but which would later lead to aggregate output growth at a rate higher than the one achievable along the competitive equilibrium. This could happen if fixed costs, in the form of a minimum scale of operation, were assumed, but cannot happen under costant returns to scale. We will discuss this fact later when we examine the role of initial conditions in growth.

Theorem 4 is still deficient, however, in that it refers only to either total output (GNP + the market value of the stock of capital left after depreciation) or NNP. Since, to state an analogous result about GNP we would have to introduce arbitrary conventions about depreciation, the most direct remedy is to study consumption directly. Obviously Theorem 4 cannot hold for consumption rather than output: it would always be possible to consume a great deal in a single period by diverting production out of the capital sector(s). The next theorem shows, however, that the only way to consume more than the socially optimal plan is either to invest more than is socially optimal, or periodically divert production into activities that generate consumption. In particular, if the investment rate is fixed, then no plan can have a higher rate of consumption growth than the competitive equilibrium plan.

Let χ denote the characteristic function taking the value 1 if a condition is true, and 0 otherwise.

Theorem 4.4. *If a solution to the social planner problem λ^*, k^*, x^* is supported by prices p^* and a socially feasible plan λ, k, x satisfies*

$$\frac{p_t^* k_t}{p_t^*(k_t + x_t)} \leq \frac{p_t^* k_t^*}{p_t^*(k_t^* + x_t^*)}$$

then

$$\limsup_{T\to\infty} \frac{\sum_{t=1}^{T} \chi(p_t^* x_t > \beta p_t^* x_t^*)}{T} < \frac{1}{\beta}$$

What this says is that the frequency with which the value of consumption under the alternative plan x exceeds the socially optimal value of consumption by a factor of β goes to zero as $\beta \to \infty$. It also implies that the frequency with which it exceeds the socially optimal value of consumption at all, is less than 1.

We mentioned earlier that in Theorem 3 the assumption that all valuable output can be diverted into next period capital is necessary for the conclusion. We give now a counterexample showing that this is the case.

4.1. **Example - Joint Production.** We start with the simple vintage capital model of example 3.1. Recall that there were 2 sequences of activities: one producing ρ units of new vintage capital from old vintage capital, and one producing $gamma^i$ units of consumption from one unit of vintage i capital. Preferences were CES, and we found in Proposition 1 that consumption of the characteristic grows at the rate $g_c = (\delta\rho\gamma)^{1/(1+\theta)}$ provided that $\delta g_c^{-\theta} < 1$. Moreover, this growth rate required that a constant fraction

$$\phi = 1 - (\frac{1}{\rho})(\delta\rho\gamma^{-\theta})^{1/(1+\theta)}$$

of the stock of current vintage capital be used in the production of the consumption good. Another way to say this is to say that 1 unit of capital of vintage i is used in equilibrium to produce $(1-\phi)\rho$ units of next period capital and $\phi\gamma^i$ units of consumption. Suppose that we add a second type of vintage capital that can be used only in a single sequence of activities, with one unit of vintage i of the second type of capital jointly producing $(1-\tilde{\phi})\tilde{\rho}$ units of the next vintage of the same type of capital and $\tilde{\phi}\gamma^i$ units of consumption. Suppose also that there is a single unit of initial capital that may be freely converted into either type 1 or type 2 capital (in any proportion).

Suppose that $\tilde{\rho} > \rho$ and that $\phi = \tilde{\phi}$. Then the second technology is the only one that will ever be used: for the same capital set-aside used by the first technology in equilibrium, the second technology simply grows faster. On the other hand, as long as $(1-\tilde{\phi})\tilde{\rho} < \rho$ the first technology dominates the second in terms of *potential* growth rates: under the first technology, if a sufficiently large fraction of the capital stock is diverted to the capital sector, the economy can grow at a rate arbitrarily close to ρ, which is impossible when the second one is used.

4.2. **Example - Pricing New Commodities:** Consider an economy in which there are two characteristics, so $J = 2$, one agent, so $H = 1$, three consumption commodities, three kinds of capital and one type of labor. The utility function over the two characteristics is a symmetric Cobb Douglas, i.e. $u(c_1, c_2) = (c_1)^{1/2}(c_2)^{1/2}$. The three consumption goods have the following vectors of characteristics:

$$C_1 = [1,0] \qquad C_2 = [0,1] \qquad C_3 = [\varepsilon, 1]; \qquad \varepsilon > 0$$

Write the commodity vector as $x = [z_1, z_2, z_3, \ell, \kappa_1, \kappa_2, \kappa_3,]$. To economize on notation let χ_i^κ and χ_i^z denote, respectively, the three-dimensional vectors with one in the position of capital good i and zero elsewhere, and one unit in the position of the consumption good i and zero elsewhere. There are six potential activities, one for each consumption good and one for each capital good. They are

$$a_{z_i} = [0,0,\chi_i^\kappa;\chi_i^z,0,0] \qquad i = 1,2,3.$$
$$a_{\kappa_i} = [0,1,0;0,1,\chi_i^\kappa] \qquad i = 1,2,3.$$

In words: capital of type i produces, on a 1-1 basis, the relative consumption good; labor can produce any of the three capital goods, again on a 1-1 basis while also reproducing itself.

Now assume this economy begins with and endowment of 2 units of labor and 1 unit each of κ_1 and κ_2. The set of initially available activities is $A_0 = \{a_{z_1}, a_{z_2}, a_{\kappa_1}, a_{\kappa_2}\}$. So, at the beginning, the third consumption good is not viable, nor is the tool producing it.

As long as $A_t = A_0$ the optimal production plan is

$$\lambda(a_{z_1}) = \quad \lambda(a_{z_2}) = \quad \lambda(a_{\kappa_1}) = \quad \lambda(a_{\kappa_2}) = 1.$$

The supporting prices for the two consumption goods are

$$p_{z_1 t+1} = p_{z_2 t+1} = \frac{\delta^t}{2}, \qquad t = 0,1,\ldots.$$

The zero profit condition can be applied first to a_{z_1}, a_{z_2} and then to $a_{\kappa_1}, a_{\kappa_2}$ to derive, in turn, the equilibrium prices of capital and labor

$$p_{\kappa_i t} = p_{z_i t+1} = \frac{\delta^t}{2}, \quad i = 1,2; \quad t = 0,1,\ldots.$$
$$w_{t+1} = w_t - p_{\kappa_i t+1}, \quad i = 1,2; \quad t = 0,1,\ldots.$$

with w_0 being determined by the intertemporal budget constraint. Now consider what happens when the third pair of goods and the two activities through which they can be produced are discovered. Our interest here is not in the transition paths and the oscillations in the value of aggregate activity they can bring about (for this, see below Example 7.1). Hence we will look directly at the new steady state. Let $A_T = \{a_{z_1}, a_{z_2}, a_{z_3} a_{\kappa_1}, a_{\kappa_2}, a_{\kappa_3}\}$. There are still only two units of labor available, which implies that, in total, at most two units of the three consumption goods can be produced. In equilibrium, we will have $z_2 = 0$ as the third consumption good costs as much labor as the second but provides a strictly greater vector of characteristics. Hence, after the transition period, $\kappa_2 = 0$ and the two units of labor are allocated to the production of κ_1 and κ_3. Maximization of steady state utility gives the optimal production plan. Along this, an amount equal to 1 of the first characteristic and an amount equal to $1/(1-\varepsilon)$ of the second are produced and consumed in each period $t = T+1, T+2, \ldots$

$$\lambda(a_{z_1}) = \lambda(a_{\kappa_1}) = \frac{1-2\varepsilon}{1-\varepsilon},$$
$$\lambda(a_{z_2}) = \lambda(a_{\kappa_2}) = 0,$$
$$\lambda(a_{z_3}) = \lambda(a_{\kappa_3}) = \frac{1}{1-\varepsilon}.$$

The supporting prices for the three consumption goods, the three capital goods and the labor input can be computed once again by straightforward differentiation and application of the zero profit condition. Write $\eta = (1-\varepsilon)^{1/2} < 1$. Then:

$$p_{z_1 t+1} = p_{z_3 t+1} = \frac{\delta^t}{2\eta}, \qquad t = T, T+1, \ldots.$$
$$p_{z_2 t+1} = \frac{\delta^t \eta}{2}, \qquad t = T, T+1, \ldots$$
$$p_{\kappa_i t} = \frac{\delta^t}{2\eta}, \quad i = 1,2,3; \quad t = T, T+1, \ldots.$$
$$w_{t+1} = w_t - p_{\kappa_i t+1}, \quad i = 1,2,3; \quad t = T, T+1, \ldots.$$

with w_0 determined again by the intertemporal budget constraint.At the new equilibrium prices the activity a_{z_2} makes negative profits

$$\pi_t^{z_2} = \frac{\delta^t \eta}{2} - \frac{\delta^t}{2\eta} < 0, \quad \text{as } \eta < 1;$$

which justifies the choice of $\lambda(a_{z_2}) = 0$. Notice also that

$$p_{z_3 t+1} = \varepsilon \cdot (p_{z_1 t+1}) + 1 \cdot (p_{z_2 t+1}),$$

that is to say that the price of the third consumption good is a linear combination of the supporting prices of the other two goods, with weights equal to the coordinates of its characteristics vector. This "pricing by arbitrage" of the new goods can be given a formulation that goes beyond this particular example.

Theorem 4.5. *Let λ^*, k^*, x^* be a competitive equilibrium supported by the price sequence p^*. Consider a period $t \in \{1,2,\dots\}$ and a good n which is viable at t. Let C_n be its characteristic vector and denote with p^*_{nt} the price of this good at time t. Assume there exists a collection of goods $\{n_1,\dots,n_{n'}\}$, which are viable at time t and have characteristic vectors C_{n_j} such that*

$$C_n = \sum_{j=1}^{n'} \alpha_j \cdot C_{n_j}, \qquad \alpha_j \in \Re \, \forall j.$$

Then

$$p^*_{n,t} = \sum_{j=1}^{n'} \alpha_j \cdot p^*_{n_j,t}.$$

4.3. **Example - Economic Rents and Their Dissipation:** The last example is a good starting point for discussing the way in which entrepreneurial innovation generates economic rents and how they are eliminated or not in competitive equilibrium.

The assumption that there is only one type of labor, which is equally effective in producing any of the three kinds of capital, implies that the "potential rents" generated by the invention of the third consumption good cannot be appropriated by labor itself. That is to say, they are immediately transferred to the final consumer in the form of higher consumption of the second characteristic and, consequently, higher utility.

Nothing is retained (in the form of a higher price) by the factors of production: labor employed in a_{κ_3} is perfectly substitutable with labor employed in a_{κ_1}, hence they earn the same wage rate. This implies, in turn, that the two capital goods are also equally priced. Consider now the following, slightly different, situation. Instead of one kind of labor we have two kinds of labor supplied in equal amounts, $\ell^1 = 1$ and $\ell^2 = 1$. The difference between ℓ^1 and ℓ^2 is that only the latter is able to produce κ_3. Hence ℓ^2 can be used in any of the three activities $a_{\kappa_1}, a_{\kappa_2}, a_{\kappa_3}$ with unchanged productivity, while usage of ℓ^1 is limited to the first two.

The competitive equilibrium when $A_t = \{a_{z_1}, a_{z_2}, a_{\kappa_1}, a_{\kappa_2}\}$ is unchanged from before: the two inputs are perfect substitutes, given the viable technology set, and therefore earn the same income.

Things become quite different when z_3 and κ_3 are introduced and $A_t = \{a_{z_1}, a_{z_2}, a_{z_3} a_{\kappa_1}, a_{\kappa_2}, a_{\kappa_3}\}$. The equilibrium allocation becomes

$$\tilde\lambda(a_{z_1}) = \tilde\lambda(a_{\kappa_1}) = 1 - \frac{\varepsilon}{2}$$

$$\tilde\lambda(a_{z_2}) = \tilde\lambda(a_{\kappa_2}) = \frac{\varepsilon}{2},$$

$$\tilde\lambda(a_{z_3}) = \tilde\lambda(a_{\kappa_3}) = 1$$

so that the equilibrium consumption of the two characteristics is $\tilde{c}_i = 1 + \frac{\varepsilon}{2}$ for $i = 1,2$. The new supporting prices therefore are

$$\tilde{p}_{z_1 t+1} = \tilde{p}_{\kappa_1 t} = \tilde{p}_{z_2 t+1} = \tilde{p}_{\kappa_2 t} = \frac{\delta^t}{2},$$

and

$$\tilde{p}_{z_3 t+1} = \tilde{p}_{\kappa_3 t} = \frac{\delta^t (1+\varepsilon)}{2}$$

for all $t = T, T+1, \ldots$. Notice that Theorem 6 still holds, as

$$\tilde{p}_{z_3 t+1} = \varepsilon \cdot (\tilde{p}_{z_1 t+1}) + 1 \cdot (\tilde{p}_{z_2 t+1}).$$

Contrary to the previous case in which all inputs were perfect substitutes, the introduction of the new good now generates a "rent" which goes to the input producing the new commodity. The total value of this rent is large, as it corresponds to the total increase in aggregate output, i.e. $\frac{\varepsilon \delta^t}{2}$.

On the other hand this rent is not "too large", at least if we look at it from the point of view of the incentives to innovate. Once the new activities are discovered the difference between total output "with" ℓ^2 and "without" ℓ^2 is exactly equal to the increase in ℓ^2 total income. In the language of Makovski and Ostroy [1995], this corresponds to "full appropriation" and generates the correct incentives for implementing the innovation. It is also interesting to note how the redistributional impact of innovative activity comes about by means of relative prices. To do this, compare the "post-innovation" prices of the three consumption goods for the case in which all inputs are perfect substitutes with the current one. As expected we have

$$\tilde{p}_{z_1 t+1} < p_{z_1 t+1}; \qquad \text{and} \qquad \tilde{p}_{z_2 t+1} > p_{z_2 t+1};$$

which explains why the two activities a_{z_2} and a_{κ_2} are now operated at a positive level. On the other hand one can check the following

$$\tilde{p}_{z_3 t+1} > p_{z_3 t+1} \qquad \text{if } \varepsilon \in [0, \frac{1}{2}(\sqrt{5}-1))$$

and

$$\tilde{p}_{z_3 t+1} \leq p_{z_3 t+1} \qquad \text{if } \varepsilon \in [\frac{1}{2}(\sqrt{5}-1)]$$

5. Endogenous Growth

We now study growth and the introduction of new technologies through a series of examples.

5.1. **Example - Endogenous Growth.** Modify example 3.1 to allow the possibility of producing the existing vintage of capital. We continue to assume a single characteristic and a single consumer; that the period utility is CES; that there is a single consumption good and a sequence of different vintages of capital. We retain the two types of technologies described above and an initial endowment of 1 unit of vintage zero capital. In addition we assume that it is also possible to produce the current vintage of capital by means of the activity $[0, \chi_i; 0, \beta\chi_i]$ $\beta > 0$. This means that it is optional rather than mandatory to move on to the next vintage. Moreover we assume that $\beta > \rho$ so that a unit of current capital is more productive at reproducing itself than at producing the next vintage.

Proposition 5.1. *If* $\gamma\rho > \beta$ *then the only vintage of capital produced is the latest possible vintage*

Proof. One unit of current capital can be used either to produce ρ units of new capital or β units of old capital. Next period ρ units of new capital produce $\gamma^{t+1}\rho$ units of consumption while β units of old capital produce $\gamma^t\beta$ units of the same. If the parametric restriction is satisfied production of the new capital dominates production of the old capital. □

In this case the endogenous growth reduces to the simple vintage capital case of example 3.1.

This same way of modelling improvements in the productivity of capital, can be used to give the simplest model in which the existence of a finite amount of natural resources which are necessary for production, need not create a binding constraint upon economic growth.

5.2. **Example - Non-Binding Natural Resources.** Introduce a natural resource, say labor, in the previous economy so that the commodity vector is now written (z, κ, ℓ).

The activities that produce consumption by means of capital and labor have the form $[0, \chi_i, 1; \gamma^i, 0, 1]$ $\gamma > 1$, those producing old and new capital become, respectively, $[0, \chi_i, 1; 0, \beta\chi_i, 1]$ $\beta > 1$, and $[0, \chi_i, 1; 0, \rho\chi_{i+1}, 1]$ $\rho > 1$. Notice that each activity, in addition to producing the primary output, reproduces the labor input in the current period as labor output in the following period. We also add a fourth type of activity that produces next period labor by means of current period labor $[0, 0, 1; 0, 0, 1]$. The latter activity will be operated to maintain a constant labor force in those circumstances in which the initial allocation of capital makes part of the available labor force redundant in the other production activities. Endowments are one unit of vintage 0 capital stock and one unit of labor ℓ.

Again, what matters is that $\gamma\rho > \beta$, and the result is exactly as in the previous example: only capital of the most recent vintage will be produced.

As before, let κ_{tt} denote the amount of vintage t capital at time t. Notice that $\kappa_{tt} \leq 1$ can be assumed for all $t \geq 1$. Assume $\kappa_{tt} < 1$, as before we search for social optima along which a constant fraction ϕ of capital is allocated to the production of the consumption commodity. One extra unit of this capital can be used together with one extra unit of labor to produce γ^t units of consumption good, yielding a marginal present value utility of $\delta^t\gamma^t(\phi\gamma^t\kappa_{tt})^{-\theta-1}$. Alternatively the same mix of capital and labor can be used to produce ρ units of vintage $t+1$ capital for time $t+1$, leading to a present value marginal utility of $\rho\delta^{t+1}\gamma^{t+1}(\phi\gamma^{t+1}\kappa_{t+1,t+1})$, while $\kappa_{t+1,t+1} = \rho(1-\phi)\kappa_{tt}$. Equating payoffs from the two alternative uses of capital we get

$$1 = \delta(\gamma\rho)(1-\phi)$$

which is exactly the restriction we obtained in the simple vintage capital model. Hence the physical stock of capital will be growing at a rate $g_\kappa = (\delta\rho\gamma^{-\theta})^{1/(1+\theta)}$ until it reaches $\kappa_{TT} = 1$ at some finite date T. Actual growth requires $g_\kappa > 1$, that is to say

$$\delta\gamma^{-\theta} > \rho^{-1},$$

otherwise it will be convenient to let the stock of physical capital slowly shrink to zero.

The latter inequality also guarantees that, after period $t = T$, the physical amount of capital will remain constant at one. Increasing it further is clearly not optimal, as all the available natural resource is already fully employed. In this circumstances, a fraction $\tilde{\phi} = (\rho-1)/\rho$ of the mixed capital–labor input is allocated to the production of consumption and labor, and the remaining $1/\rho$ portion goes into the production of a unit of new vintage capital and $1/\rho$ units of labor.

Use the same technique as before to compute the supporting prices for all periods $t > T$ in which the capital stock is kept constant at one. From $p^*_{nt} \geq \mu^h\delta^{t-1}Du^h(Cx^{h*}_t)C_n$we get p^*_{zt}, the price of the consumption good:

$$p^*_{z,t+1} = \tilde{\phi}(\delta(\gamma)^{-\theta-1})^t$$

Where $\tilde{\phi}$ is a positive constant. Denote with w_t the price of labor. The zero profit condition for the consumption activity is

$$p^*_{z,t+1}\gamma^t + w_{t+1} - p^*_{\kappa_{tt}} - w_t = 0, \quad \text{or} \quad p^*_{\kappa_{tt}} = \tilde{\phi}(\delta\gamma^{-\theta})^t + w_{t+1} - w_t$$

Applying the same condition in the sector producing the new vintage capital gives us a dynamic relation for the prices of labor

$$w_{t+1} - w_t = \frac{\tilde{\phi}}{\rho}(\delta\gamma^{-\theta})^{t-1}[1 - \rho\delta\gamma^{-\theta}]$$

The aggregate budget constraint for the representative agent provides the initial value w_0. Notice that the growth condition established above, now implies that $w_{t+1} < w_t$, which imposes negative profits upon the activity which produces one unit of labor tomorrow by means of a unit of labor today. This is consistent, as this activity is not being operated when the stock of capital is equal to one.

The first order difference equation for the wages can also be used to solve explicitely for the price fo capital, which is

$$p^*_{\kappa_{tt}} = \tilde{\phi} over \rho [\delta\gamma^{-\theta}]^{t-1}$$

Finally, the transversality conditions are both satisfied as $w_t = 0$ for $t < T$ and $w_t \to 0$ as $t \to \infty$, and $p^*_{\kappa_{tt}}$ also converges to zero for all admissible parameter values.

Proposition 5.2. *After a finite number of periods, consumption of the characteristic grows at the rate $g_c = \gamma$ and the quantity of capital stock remains fixed at one from one vintage to the next provided that $\delta g_c^{-\theta} > \frac{1}{\rho}$.*

5.3. **Example - Capital Saving Innovation:** The form of technological progress considered in the previous example is generally termed "neutral": it simply shifts up the quantity of consumption good that may be obtained from a given combination of inputs. The literature on technological progress has also made a distinction between capital- and labor-saving innovations. Holding output and the amount of the other factor fixed, they bring about, respectively, a reduction in the input requirement of capital and labor.

One can look at our vintage capital model as the most simple example of capital-saving technical progress when only reproducible factors are ultimately used in production. In these circumstances capital-saving innovations allow for unbounded growth in the consumption of characteristics without the need for accumulation of an infinite amount of capital. On the other hand when a non-reproducible factor is also used in production and its substitutability is limited, capital-saving technological progress alone will not make possible unbounded growth in the consumption of characteristics.

To see this, one only needs to modify our last example assuming that the technology for the production of the consumption good takes the form $[0, \chi_i/(\gamma^i), 1; 1, 0, 1]$ $\gamma > 1$ while everything else remains unaltered. Along the competitive equilibrium path aggregate consumption is bounded above by one. New vintages of the capital stock are still introduced but in smaller and smaller quantities. Asymptotically the economy produces the same unit of the consumption good by using an infinitesimal amount of the last vintage capital.

Things become completely different when the labor input requirement is dropped from the consumption activity. As long as $\rho\gamma > \beta$, it is socially optimal to introduce the new vintage capital in each period. Hence the output of the consumption sector increases over time. The amount of new vintage capital produced may be either growing (toward the upper bound of ρ units) or asymptotically shrinking to zero depending upon $\rho\delta\gamma^{theta} > (<)1$.

5.4. **Example - Labor Saving Innovation.** In the next example we look at the equilibrium implications of labor saving innovations. The improved capital vintages, rather than increasing output per unit of capital, reduce the labor requirement for one unit of output. As a result, the economy can grow, but only by moving on to more advanced vintages of capital that make it possible to produce increased amounts of output from the existing stock of labor. The commodity space is as in the previous two examples and we maintain the activity producing next period labor from the current one.

The activities producing the two capitals do not require labor inputs, i.e. $[0,\chi_i,0;0,\beta\chi_i,0]$ $\beta > 0$ and $[0,\chi_i,0;0,\rho\chi_{i+1},0]$ $\rho > 0$. The activity for producing consumption is $[0,\chi_i,1/\gamma^i;1,0,1/\gamma^i]$ $\gamma > 0$ so that the benefit of better vintage capital is that it reduces the amount of labor input required to produce a unit of consumption. Endowments are one unit of vintage 0 capital and one unit of labor. We consider two different situations.

Case 1: Natural resources are never binding

This is possible only if the new vintage capitals are continuously introduced and, in such an equilibrium, the growth rate of the physical stock never exceeds γ. These circumstances are most easily realized when only the next vintage of capital can be produced (as in Example 3.1) and the current vintage cannot be reproduced, which we assume to ease the exposition. Recall that the labor stock remains constant at a single unit.

If, at time t κ_{tt} units of vintage t capital are available, the maximum producible amount of consumption is $\min\{\gamma^t,\phi_t\kappa_{tt}\}$ where ϕ_t is the share of capital stock allocated to the consumption sector in period t. One can show that there exists a unique equilibrium path, along which $\phi_t \to \phi$ and the stock of capital grows, eventually, at a constant rate g_κ. Therefore we set $\phi_t = \phi$ and study this particular form of balanced growth.

If the labor constraint does not bind, i.e. if $[(\rho-1)/\rho]\kappa_{tt} \leq \gamma^t$, the analysis becomes identical to example 3.1, except that $\gamma = 1$ should be replaced in all the growth-rate formulas. In other words, the growth rate of consumption is $g_c = (\delta\rho)^{1/(1+\theta)}$ provided the transversality condition $\delta g_c^{-\theta} < 1$ is satisfied. Prices are given by $p^*_{z,t} \propto \rho^{-t}$ and $p^*_{\kappa_{tt}} \propto \rho^{-t}$, and labor has a price of zero. Moreover, the growth rate of the capital stock g_κ is the same as the growth rate of consumption.

This, in turn, implies that the labor constraint will not bind (except possibly in early peiods) provided that this growth is smaller than γ, which is the rate at which the labor requirement is reduced; that is $\gamma > (\delta\rho)^{1/(1+\theta)}$.

Case 2: Natural resources are binding

We assume here that the initial amount of capital k_0 of type 0 is less than one (the total and fixed amount of natural resource). The initial stages of growth will entail a (finite) number of periods at which the economy grows at the unconstrained rate, which is larger than γ. After T periods, g_κ becomes zero as the stock of capital (measured in phisical units) stays constant at one and consumption grows only because the new capital goods are more efficient, at producing consumption, than the old ones. Hence consumption grows at a rate γ per period. The proportion of capital allocated to the investment sector is obviously $1/\rho$.

Claim: in this case no innovation until the capital stock becomes large enough (relative to labor endowment). We should be able to show the following pattern occurs in equiibrium. Build up capital of the last vintage until labor becomes a constraint, then innovate by introducing the next vintage, build up its stock again and exhaust the stock of the previous vintage. Move on to the next vintage only when the labor constraint becomes binding again, etc. In equilibrium at most two (consecutive) vintages of capital are being used.

6. Sources of Growth

We now investigate how some of the forces that are commonly believed to lead to growth operates in our model.

Initial endowments can be important insofar as they prevent the use of certain technologies. However, broadly speaking, two economies that have the same set of viable activities will, in the long-run, use technologies with the same growth potential. On the other hand, while long-run growth may be the same in the two countries, the length of time before a particular technology becomes dominant may depend on initial conditions. In addition, history may also make a difference in the types of characteristics that are produced in the long run. In one of the following examples, we find that in two economies with the same growth potential and the same long run growth rates, the long run characteristic intensity is quite different.

Initial conditions may also be relevant in determining the speed with which different countries may adopt new technologies. A more advanced country, i.e. one in which initial conditions allowed either faster accumulation of capital or an earlier adoption of less costly methods of production, may find it not convenient to adopt an even more productive technology at a certain point in time. The new technology may instead be more rapidly adopted by a less advanced country which will therefore grow faster for a certain amount of time.

Another force that may lead to sustanined economic development, is increase in international trade. It is a rather straightforward exercise to build special cases of our model in which trade is beneficial to long run growth in all countries. This may occurr, again, in the presence of unequal initial conditions accross countries, when the set of viable activities is not the same or when saving propensities are different. On the other hand, as long as the hypothesis of constant returns to scale is maintained, our model suggests that countries that have the same set of initial conditions, the same set of viable activities and the same saving propensities, will not engineer a permanent increase in growth rates by trading with each other, even if they have different preferences over the set of characteristics.

The expansion of commerce is often perceived as the force that leads to an increase in the division of labor, which is seen, in turn, as the true source of economic development. One would argue, in fact, that it is the division of labor, and not trade per-se, that makes the adoption of more efficient production methods advantageous. In our view this phenomenon can be properly understood only by assuming the existence of fixed costs and a minimum scale of operation for certain activities. The extent to which fixed costs can be made consistent with competitive equilibrium is an open question, the relevance of which is underlied by our analysis.

Our model also shows that, within a competitive equilibrium framework, differences in income distribution have an impact more like that of initial conditions: varying the income distribution can delay the dominance of a particular technology, but will not generally effect the long-term prospects for growth as long as fixed costs are not present.

6.1. **Initial Conditions.** We begin by considering two economies that differ only in their initial endowments and ask how this impacts in the long-run growth and development of the economy. Obviously if a sequence of technologies that yields an especially high growth rate requires a particular type of commodity to get started, an economy that begins without this commodity cannot use that technology, and may grow more slowly than an economy for which this sequence is viable. To do this, we say that two economies have *equivalent viability* if every activity viable in one economy is viable in the other.

We want to consider in particular the idea of production bottlenecks: that a superior technology may fail to be introduced because the initial start up cost is very high. For example, it has been argued that the reason that the industrial revolution began in Europe rather than in China, despite the fact that much of the technology (steel, explosives, etc.) that triggered the industrial revolution orginated in China, is because cheap coal was available in Europe but not China.

Consider Example 5.1 in which either new vintages of capital can be produced at rate ρ, or old vintages at rate β. Suppose that the second vintage of capital may not be produced from existing capital, but from one of two types of capital, one of which may also be used to produce a large amount of consumption. It can be argued that an economy in which only this second type of capital is present may fail to introduce the second vintage (and, consequently, subsequent vintages) of capital because to do so requires the sacrifice of a very large amount of consumption. Again, if we had a nonconvex technology in which activities may not be operated below a certain fixed level, it would be easy to construct examples in which this is the case. However, Theorems 2–5 show that this cannot be the case under our maintained assumption that every activity displays constant returns to scale. If we focus not on long-term growth rates, but rather on the nature of the long-run economy, it is easier to produce examples of history dependence. The next example shows how the relative scarcity of different characteristics can depend upon the initial conditions. The following one, based on an argument by Jones and Ohiyama [1994], shows that less developed countries may have a comparative advantage at introducing new technologies and this may enable them to catch up with the leaders.

Example 6.1–History Dependence with Two Characteristics:

We start with the basic setup of Example 3.1 in which there are no fixed factors, and existing capital of vintage i could either produce the consumption commodity or capital of vintage $i+1$. However, we now suppose that there are two characteristics, so that $J=2$, and there will also be two consumption commodities and two kinds of capital for each vintage i. The period utility function is $u(c) = -(1/\theta)[(c_1)^{-\theta} + (c_2)^{-\theta}]$. The first consumption commodity has a vector of characteristics $C_1 = [A,B]$ and the second has $C_2 = [B,A]$, with $B < A$.

We write a commodity vector as $x = (z_1, \kappa_1, z_2, \kappa_2)$ and we let χ_i^j be the vector consisting of a single unit of type $j = 1,2$ capital of vintage $i = 0,1,\ldots,$, and zero units of all other commodities.

There are six types of activities. One type of activity $[0,\chi_i^1,0,0;\gamma^i,0,0,0]$ produces the first consumption commodity from type 1 capital of vintage i. The second type of activity $[0,\chi_i^1,0,0;0,\rho\chi_{i+1}^1,0,0]$ produces the new vintage of type one capital from the old vintage of type one capital. The third type of activity $[0,0,0,\chi_i^2;0,0,\gamma^i,0]$ produces the second consumption commodity from the second type of capital of vintage i. The fourth type of activity $[0,0,0,\chi_i^2;0,0,0,\rho\chi_{i+1}^2]$ produces the new vintage of type two capital from the old vintage of type two capital. The fifth type of activity

$[0,\chi_i^1,0,0;0,0,0,\xi^i\chi_0^2]$ converts type 1 capital of vintage i into ξ^i units of type 2 capital of vintage 0. The sixth type of activity $[0,0,0,\chi_i^2;0,\xi^i\chi_0^1,0,0]$ operates in the reverse direction.

Proposition 6.1. *Assume the initial endowment is a single unit of type* 1 *capital of some vintage* i_0*. If*

$$i_0+1 > \frac{\log(A\xi^i/B\rho)}{\log\gamma}$$

only type 1 capital is ever produced.

Proof. Consider the production of characteristic 2 from the viewpoint of the initial period; 1 unit of type 1 capital of the ith vintage can produce ρ units of type 1 vintage $i+1$ capital which in turn can produce $\rho\gamma^{i+1}B$ units of characteristic 2 the period after (i.e. two periods from $t=0$). Alternatively, ξ^i units of type 2 vintage 0 capital can be produced using the same unit of type 1 vintage i. In two periods time this could produce ξ^iA units of characteristic 2. Consequently, in all future periods the ratio between the amount of characteristic 2 produced by continuing with the same technology to that achievable from switching technologies is $\rho\gamma^{i+1}B/\xi^iA$. □

What this means is that, in the long-run, the ratio between the total amount of characteristic 1 and characteristic 2 consumed depends on the initial condition of the economy, even though the different initial conditions have equivalent viability.

6.2. **Example - Coming from Behind.** We use a simplifies version of our work-horse, the vintage capital model. As in example 3.1, assume there exists only one agent and one characteristic, and there are no fixed factors. There are three kinds of commodities: the consumption good, which provides the characteristic on a one-to-one basis, and two kinds of capital goods, each of which comes in different vintages. Denote these two kinds of capital goods as κ^1 and κ^2 and write a commodity vector as $x=[z,\kappa^1,\kappa^2]$. We still use the indicator function χ_i to represent a unit of vintage i capital stock; the position of χ_i in the commodity vector will tell if it is of type one or type two. So for, example, $[0,0,\chi_3]$ is a vector with zero consumption, zero amount of κ^1 and one unit of κ^2 of the 3rd vintage.

The whole set of activities is composed by the following triplets, for $i=0,1,\ldots$

$$a_{\kappa_i^1} = [0,\chi_i,0;\gamma^i,\chi_{i+1},0], \quad \gamma>1;$$

$$a_{\kappa_i^2} = [0,0,\chi_i;\mu^i,0,\chi_{i+1}] \quad \mu>1;$$

$$a_{\kappa_i^{12}} = [0,\chi_i,0;0,0,\beta\chi_1] \quad \beta>0.$$

We assume furthermore that $\mu > \gamma$ so that κ^2 is a better kind of capital stock, as it can produce its next vintage to the same extent of κ^1 but always yields more of the consumption good.

We assume that at time $t = 0$ the set of viable technologies is $A_0 = (\{a_{\kappa_i^1}\}_{i=0}^{\infty})$ and the initial endowment is one unit of vintage zero κ^1. As long as no innovation takes place the equilibrium allocation and prices are staightforward. Assuming the utility function is a CES, we have, for $t = 0, 1, \ldots$

$$c_{t+1} = \gamma^t, \quad \kappa^1_{tt} = 1;$$

$$p^z_{t+1} = (\frac{\delta}{\gamma^{1+\theta}}), \quad p^{\kappa^1_t}_{t+1} = p^{\kappa^1_{t-1}}_t - p^z_{t+1}$$

Setting $\delta < \gamma^\theta$ and $p_0^{\kappa_0^1} = \frac{\gamma^{1+\theta}}{\gamma^{1+\theta}-\delta}$ guarantees that utility is bounded and the transversality condition is satisfied. ßAssume now that, at time $t = T$, κ^2 becomes known and the set of viable activities expands to $A_t = (\{a_{\kappa_i^1}\}_{i=0}^{\infty}, \{a_{\kappa_i^2}\}_{i=0}^{\infty}, \{a_{\kappa_i^{12}}\}_{i=0}^{\infty})$. The question we ask is: does the speed at which the new capital stock is adopted depend upon the level of development of a country? If this is the case, which country would adopt κ^2 at a faster pace, an advanced or an underdeveloped country? Clearly this will not happen istantaneously: if the whole stock of κ^1_{TT} is invested in $a_{\kappa_T^{12}}$ it will yield β units of κ_1^2 which can produce μ units in period $T+2$ but would leave the consumer with $z_{T+1} = 0$ in the transition period.

We call a country advanced if, at time $t = T$, it is using the latest available vintage of κ^1, that is vintage T; its opportunity cost of investing a unit of κ^1_T in $a_{\kappa_T^{12}}$ is then equal to γ^T and its expected gain is equal to the whole stream of future consumption that can be produced from the β units of κ_1^2 so obtained. A less advanced country is one using some vintage $\tau < T$ and its opportunity cost of investing in $a_{\kappa_T^{12}}$ is $\gamma^\tau < \gamma^T$ while its gain are the same as those for the advanced countries. Intuition says this asymmetry should generate a "comparative advantage" for the less advanced country to use the new activity. Hence the country coming from behind will adopt the new capital stock at a faster pace than the leader.

Notice that this comparative advantage argument still holds if one assumes that the amount of κ_1^2 that can be obtained from the usage of $a_{\kappa_T^{12}}$ is not a constant β but is instead increasing in the vintage of the κ^1 capital input. Call these variable quantities β_t. What matters for the investment decision is still the ratio γ^T/β_T in one case and the ratio γ^τ/β_τ in the other.

Analytical derivation of the exact conditions for the general case is rather complicated. On one hand, we need to compute the whole sequence of marginal utilities of consumption at all future dates $T+1, T+2, \ldots$ and, on the other hand, the fraction $\phi_t \in [0,1]$ of κ^1 which is to be shifted toward a_{κ_t}[12] in each period will not be constant In order to make the intuition transparent, assume for the moment that the marginal utility of consumption is constant and consider the variational condition determining the fraction ϕ_T during the first period in which the new capital has become viable. We have that $\phi_T = 1$ (i.e. the new technology is completely and immediatey adopted) if

$$\frac{\gamma^\tau}{\beta} < \sum_{i=1}^{\infty} (\delta\mu)^i = \frac{\delta}{1-\delta\mu}$$

where $\delta\mu < 1$ has been assumed and the index $\tau \leq T$ denotes the vintage of κ^1 available in the country. The previous inequality is more likely the smaller is the index τ, i.e. the less developed is the country. In fact, large values of τ make the opposite inequality

$$\frac{\gamma^\tau}{\beta} > \sum_{i=1}^{\infty} (\delta\mu)^i = \frac{\delta}{1-\delta\mu}$$

more likely. In this case $\phi_t = 0$ for all $t \geq T$ and the very advanced country will never shift to the new technology. The argument can be easily extended to the case of a general utility function by using upper and lower bounds on the marginal utility of consumption to derive sufficient conditions for $\phi_T = 1$ or $\phi_T = 0$ to hold.

6.3. **Exchange.** We now consider the extent to which trade can facilitate growth. If we begin with two closed economies and open them to trade, the fact that the competitive equilibrium is in the core assures that the result is a Pareto improvement. More insteresting, as the next proposition argues, the open economy will never exhibit a higher growth rate than either of the two closed economies as long as the two closed economies have equal viability and equal saving propensities.

Proposition 6.2. *Consider an economy with H agents, initial condition k_0 and a sequence $\{A_t\}_{t=0}^{\infty}$ of viable activity sets. Let λ^*, k^*, x^*, p^* be a competitive equilibrium. Consider also a second economy with only 1 agent, with an initial endowment equal to ak_0, $a > 0$ and the same sequence $\{A_t\}_{t=0}^{\infty}$ of viable activity sets.. Let $\tilde{\lambda}, \tilde{k}, \tilde{x}, \tilde{p}$ be a competitive equilibrium for the second economy and assume that*

$$\frac{p_t^* k_t^*}{p_t^* y_t^*} = \frac{\tilde{p}_t \tilde{k}_t}{\tilde{p}_t \tilde{y}_t}$$

Then: $\tilde{\lambda} = a\lambda^*$, $\tilde{k} = ak^*$, $\tilde{x} = ax^*$, $\tilde{p} = p^*$ and, in the economy composed of all $H+1$ agents $(1+a)\lambda^*, (1+a)k^*, (1+a)x^*, p^*$ is a competitive equilibrium.

Proof. It is a straightforward consequence of theorems 2-4. □

7. The Dynamic Impact of Innovation

Should the process of economic innovation come to an end at a certain point in time, our model would become similar to an infinite horizon, general equilibrium economy of the kind studied in Bewley [1982], McKenzie [1986] and Yano [1984]. One difference would nevertheless remain, i.e. that feasible consumption and production sets are not bounded.

When the technology set is bounded, a number of classical theorems (see the literature quoted earlier, McKenzie [1968, 1976] and Scheinkman [1976]) can be used to show that, independently from the initial aggregate stock and, in certain circumstances, also independently of its allocation among the H households, when the discount factor δ is close enough to one, the equilibrium trajectories converge to a unique stationary state in which the stock of capital and the consumption vectors remain constant forever.

In the context of our model it is reasonabe to conjecture that a similar kind of result must be true with respect to balanced growth paths, i.e. that when the households are sufficiently patient the long-run growth rate of capital and consumption converges to some constant value unless innovative activity takes place.

To do this one needs to first transform the competitive equilibrium into a optimal growth problem and then prove a turnpike theorem for such optimal growth problem. If we are interested in competitive equilibria with transfersMcKenzie [1995] points out and solves a number of technical difficulties one needs to face in defining the welfare weights when the optimal growth model is derived from a competitive equilibrium without transfers. It is enough to pick a vector of weights $\mu = (\mu_1, \dots, \mu_H) \geq 0$ and define the aggregate utility function $U(c) = \max c^1, \dots, c^H \{\sum_{h=1}^{H} \mu^h u^h(c^h) | \sum_{h=1}^{H} c^h = c\}$, and write the optimal growth problem as

$$\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=1}^{\infty} U(c_t)\delta^{t-1}$$

$$\text{subject to: } c_t = C \cdot x_t;\ \ (k_t, k_{t+1} + x_{t+1}) \in Y;\ k_0 \text{ given}$$

Where the set Y is obtained from the set of available activities A in the natural way.

An optimal balanced growth path for this model is then defined by the pair $(\alpha^*_\delta, \lambda^*_\delta) \in \Re_+ \times \Re^A_+$ satisfying

$$(\alpha^*_\delta, \lambda^*_\delta) \in \arg\max_{\alpha,\lambda} \{ \frac{U(\alpha \cdot Cx)}{U(Cx)} \ | \ x = \sum_{a \in A} \lambda(a)[y(a) - k(a)]\}$$

The optimal balanced production plan is then obtained as

$$\bullet y^*_{t+1} = (\alpha^*_\delta)^t \sum_{a\in A} \lambda^*_\delta(a) y(a) = \alpha^*_\delta y^*_t$$

$$\bullet k^*_{t+1} = (\alpha^*_\delta)^t \sum_{a\in A} \lambda^*_\delta(a) k(a) = \alpha^*_\delta k^*_t$$

Theorem 7.1. *Suppose $A_t = A_T = A$ for all $t \geq T$ and assume the period utilities $u^h(\cdot)$ are homogeneous of degree $\theta \in (-\infty, 1)$. Then there exists a $0 < \delta^* < 1$ such that, for all $\delta^* \leq \delta < 1$ and for all k_0 such that $A \subset A_T(k_0)$, there exists a pair $\alpha^*_\delta, \lambda^*_\delta)$ with the following property*

- $\forall \varepsilon > 0$ there exists a $\tau_\delta(\varepsilon)$ such that the optimal production plan $(\alpha_\delta, \lambda_\delta)$ does not stays more than $\tau_\delta(\varepsilon)$ periods outside an ε neighborhood of $\alpha^*_\delta, \lambda^*_\delta)$.

On the other hand, it is in the spirit of our model that technological innovations and their economic profitability are somewhat unbounded. Hence one would expect a typical equilibrium path to exhibit oscillations in growth rates, with large variations around periods in which innovative activity is high and some form of partial convergence to steady growth when innovations are few or none. The overall sequences of output and consumption will therefore not evolve at some constant rate of balanced growth but display, instead, persistent oscillations in growth rates between some upper and lower bound determined by the sequence A_t, the initial stock k_0 and preferences $(u^1, \dots, u^H), \delta$.

In particular, while it is obvious that for two set of activities $A_t, A_{t'}$ with $t' > t$ our assumptions imply that $\alpha^*_\delta(A_t) \leq \alpha^*_\delta(A_{t'})$, this does not imply that, in the presence of economic innovation, observed growth rates of aggregate capital, output and even consumption should display any form of monotonicity.

In other words, our model is consistent with endogenous oscillations of aggregate output, investment and consumption, as the next example shows.

Example 7.1–Innovation and the Business Cycle

In this example there is only one characteristic, so that $J = 1$ and a single agent, so that $H = 1$. The period utility function is $u(c) = -(1/\theta)(c)^{-\theta}$. There are six xommodities, two consumption goods, two capital stocks and two kinds of labor. (Note that, to simplify notation, this is an ordinary finite economy where technological innovation is finite). The second consumption good may be converted into the characteristic on a 1-1 basis, so $C_2 = 1$, while the first consumption good only yields ε units of characteristic, so $C_1 = \varepsilon$.

We write a commodity vector as $x = (z_1, \kappa_1, \ell_1, z_2, \kappa_2, \ell_2)$. The economy is endowed with one unit of each type of labor, one unit of type 1 capital and no units of type 2 capital. To make the intuition clearer, we refer to type

1 labor as “stoop labor” and type 2 labor as “engineering”. We suppose that the first type of consumption may be produced only using as input stoop labor through the activity

$$[0,0,1,0,0,0;1,0,1,0,0,0].$$

The second type of consumption may be produced through two different activities. One uses capital and labor of the first type

$$[0,1,1,0,0,0;0,0,1,1,0,0].$$

The other activity uses capital and labor of the second type

$$[0,0,0,0,1,1/2;0,0,0,\gamma,0,1/2].$$

and it is clearly “more productive” as it requires only half unit of labor and produces $\gamma > 1$ units of consumption.

Enginnering labor, beside running the type 2 capital, is always needed to produce either type of capital. The activity producing capital of the first type is

$$[0,0,0,0,0,1;0,1,0,0,0,1].$$

while the activity producing capital of the second type has the form

$$[0,0,0,0,0,1/2;0,0,0,0,1,1/2].$$

Notice that, given the initial endowment, neither this nor the “more productive” technology to produce type 2 consumption good can be operated from the initial endowment. Indeed, we also assume that type 2 capital is completely unknown at time $t = 0$ and that it gets introduced later on, together with the activities that produce and use it. Notice also that in every activity labor exactly reproduces itself, and we assume that there are also two other activities that do nothing except reproduce labor of each type, i.e. $[0,0,1,0,0,0;0,0,1,0,0,0]$ and $[0,0,0,0,0,1;0,0,0,0,0,1]$.

In our setup and given our hypotheses on the initial endowment capital 2 is not viable. The steady state is for all labor of type 2 to be used in producing a single unit of capital of type 1 which is in turn used with one unit of labor of type 1 to produce one unit of the second type of consumption good. This could easily be made into a balanced growth path by simply assuming that the technology for producing capital of type 1 gives $\rho > 1$ units of output for each unit of input and that labor of both types could also be reproduced at some suitable rate larger than one.

In the steady state when type 2 capital has not yet been “discovered” total output is equal to one unit of consumption of type 2 and one unit of capital of type 1. The first order condition for utility maximization gives

$$p^{z_2}_{t+1} = \delta^t$$

and the zero profit conditions require:

$$p_t^{\kappa_1} + w_t^1 = p_{t+1}^{z_2} + w_{t+1}^1,$$

and

$$w_t^2 = p_{t+1}^{\kappa_1} + w_{t+1}^2$$

Hence

$$(w_{t-1}^2 - w_t^2) + (w_t^1 - w_{t+1}^1) = \delta^t$$

Assume now that an innovation occurs and that, at time $t > 0$ the set A_t comes to include all seven activities we described above. Beginning from the previous steady state this innovation generates the following sequence of events.

In period t engineers begin operating the technology producing type 2 capital, generating one unit of the same, while they also keep the activity for the "old" capital of type 1 alive by producing 1/2 unit of it. The stoop laborers operate the already existing unit of type 1 capital and produce one unit of type 2 consumption.

In period $t+1$ engineers have enough type 2 capital to abandon the production of type 1 capital whose price is expected to drop to zero at $t+2$. Hence engineers split into producing one unit of type 2 capital and operating the existing unit of type 2 capital to obtain γ units of type 2 consumption. In addition there is still half a unit of type 1 capital left, so half of the stoop laborers produce type 2 consumption and the other half type 1 consumption. Output in this period is $(\gamma + 1/2)$ units of type 2 consumption, $1/2$ unit of type 1 consumption and 1 unit of type 2 capital.

In the following $t+2$ period the economy reaches the new steady state. There is no capital of type 1 left, so all stoop laborers work producing type 1 consumption, and output is γ units of type 2 consumption, 1 unit of type 1 consumption, and 1 unit of type 2 capital.

Since$p_{nt}^* \geq \mu^h \delta^{t-1} Du^h(Cx_t^{h*})C_n$ implies that the price of type 1 consumption relative to type 2 consumption is just ε, if $\varepsilon < 1/2$ this implies that GNP falls from period $t+1$ to period $t+2$.

There are two other important features of this example we like to stress. The introduction of the new activity generates an oscillation in aggregate output which also modifies the distribution of income across households. The change in the distribution of income across households is determined by the degree of substitutability of their labor endowments in the production process and is therefore linked to the creation or destruction of "rents from innovative activity" we already discussed above.

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