Econometrica, Vol. 62, No. 2 (March, 1994), 323–342

GROWTH AND INDETERMINACY IN DYNAMIC MODELS WITH EXTERNALITIES[1]

By Michele Boldrin and Aldo Rustichini

We study the indeterminacy of equilibria in infinite horizon capital accumulation models with technological externalities. Our investigation encompasses models with bounded and unbounded accumulation paths, and models with one and two sectors of production. Under reasonable assumptions we find that equilibria are locally unique in one-sector economies. In economies with two sectors of production it is instead easy to construct examples where a positive external effect induces a two-dimensional manifold of equilibria converging to the same steady state (in the bounded case) or to the same constant growth rate (in the unbounded case). For the latter we point out that the dynamic behavior of these equilibria is quite complicated and that persistent fluctuations in their growth rates are possible.

Keywords: Externalities, uniqueness and indeterminacy of equilibrium, convergence.

1. INTRODUCTION

Our goal is to clarify the extent to which equilibria are (or are not) indeterminate in infinite horizon models of capital accumulation with a representative agent and external effects in production. We call indeterminate a situation in which there exists a continuum of distinct equilibrium paths sharing a common initial condition. In the models we study the latter is represented by the initial allocation of the capital stock.

Various models of this kind are currently being used to describe the endogenous nature of growth phenomena. Generally it is assumed that, due either to the lack of appropriate markets or to the intrinsic nature of the production process, the productivity of an individual firm's input(s), x, is affected by the aggregate level of utilization of the same or other input(s), K, so that the production function of the individual firm should be written as $f(x, K)$. In certain instances the external effect is assumed to be strong enough to induce aggregate increasing returns even if individual decision makers still face decreasing payoffs from their own inputs.

[1] This is a much revised version of a paper which was previously circulated under the title "Indeterminacy of Equilibria in Models with Infinitely Lived Agents and External Effects." We are grateful to David Cass, Leo Hurwicz, David Levine, Joe Ostroy, John Riley, and Paolo Siconolfi, and two anonymous referees of this journal for useful suggestions and criticisms. The present version benefited from comments received during presentation at the NBER General Equilibrium Group Meeting in Minneapolis (April, 1992), at the NBER Economic Growth Group Meeting in Cambridge (April, 1993) and during seminars at the Université du Quebec at Montreal, Columbia University, New York University, Brown University, University of Pennsylvania, UCLA, Stanford University, University of Tel Aviv, University of Chicago, Universidad Carlos III (Madrid), Universitat Pompeu Fabra (Barcelona), Delta-ENS (Paris), and the XI Latin American Meeting of the Econometric Society (Mexico City). The usual disclaimers apply.

Beside the obvious implication of rendering the associated competitive equilibrium inefficient the introduction of such externalities has other important effects. Here we concentrate on the positive complementarity it induces between individual actions, the full extent of which is not captured by market prices. When private returns from capital are affected by its aggregate level, multiple expectations-driven equilibria become possible. Societies with distinct institutional mechanisms may coordinate private beliefs in different ways, thereby generating different publicly held expectations about future economic events. This takes place in spite of the identical technologies, preferences, and initial economic conditions. From a theoretical viewpoint this situation is commonly described by means of dynamic models in which competitive equilibrium is indeterminate. While this need not be the only compelling explanation for the factual diversity in the growth patterns of various countries, it certainly appears as one worth investigating.

The relevance of this point of view is reinforced by the pervasiveness of indeterminacy in dynamic economic models, something of which we have started to become aware since the work of Kehoe and Levine on the Overlapping Generations Model (Kehoe-Levine (1985)). More recently a number of authors have encountered the same form of indeterminacy also in dynamic models of search and matching, e.g. Diamond-Fudenberg (1989), Howitt-McAfee (1988), Boldrin-Kiyotaki-Wright (1993), Mortensen (1991), and in dynamic models of production and accumulation when externalities are introduced, e.g. Boldrin (1992), Matsuyama (1991).

On pure logical grounds nothing seems to prevent this kind of indeterminacy from occurring also in the representative agent model of capital accumulation. Given the extent to which models of this form are now used for the purpose of empirically assessing the economic sources of growth, it is worth trying to clarify the matter. If the indeterminacy is present the interpretation of many simple estimations, obtained by pooling together data from a variety of different countries, can be questioned as there is no reason to believe that these countries should be moving along the same equilibrium path. On the other hand if a set of hypotheses can be found under which equilibria are locally unique, then one would rest assured that a minimal theoretical framework exists within which comparative static and dynamic exercises can be carried out.

For the one sector model, indeterminacy can be ruled out under fairly weak assumptions, that are consistent with those often adopted in the more applied literature. For the case of bounded accumulation, this result seems to enjoy already the status of a "folk theorem" (compare, for example, the discussion in Kehoe (1991)) and we will only briefly mention it in our exposition, without reporting the fairly obvious proof. The unbounded case is more delicate and, to the best of our knowledge, has never been examined before. We provide conditions under which all trajectories display a unique asymptotic constant growth rate, and prove that this also implies local uniqueness of equilibria. From a practical viewpoint this is tantamount to showing that the old neoclassical prediction of "convergence" emerges once again, albeit in a slightly different form. Moreover we show that under our restrictions poorer countries grow faster and growth rates are inversely correlated with income levels.

The two-sector model we examine has only one capital good, which can be interpreted either as human or physical capital. Models with both physical and human capital stocks of the kind suggested in Lucas (1988) and Romer (1990) are therefore not examined. In the absence of external effects Lucas' model has recently been studied by Caballè and Santos (1991). Also, Chamley (1992) studies an example of the same model with an externality in the accumulation of human capital. As one would expect equilibrium is unique in the world of Caballè and Santos while multiple balanced growth paths exist in the example that Chamley analyzes. Since when the first version of this paper was circulated other authors have been able to derive indeterminacy results in models of growth and externalities due to human or physical capital, most notably Benhabib and Farmer (1993), Benhabib and Perli (1993), and Xies (1993), and in dynamic models of monopolistic competition, Gali (1993).

In any case, even in our simpler world the comforting results of the one-sector framework are turned upside-down. For the two-sector model we present examples of indeterminate equilibria that are derived from very standard utility and production functions. Furthermore, in the case of unbounded growth, the same examples can exhibit indeterminate and perpetually oscillating (i.e. chaotic) asymptotic growth rates for a certain set of parameters. Quite naturally an issue of "realism" can be made with regard to the parameter values at which these more complicated phenomena arise. While they do not appear as far away from reality as those previously encountered in the optimal growth brand of the chaotic dynamics literature, they do rely on particularly strong externalities. For this reason and for the lack of reliable empirical evidence about the external effects consistent with this type of technology, we refrain from speculating on the positive implications of our findings.

As we mentioned before, the issue of indeterminacy had already been tackled for the bounded version of the one-sector growth model, e.g., Kehoe-Levine-Romer (1991), Kehoe (1991), and Spear (1991). In all three papers a one-sector growth model is studied, the difference lying in the type of external effect considered. The first specifies the individual production function as $f(x, C)$, where C is the aggregate consumption level and x is the individual stock of capital. They show by means of an example that such an economy has a locally stable steady state around which equilibria are therefore indeterminate. Kehoe (1991), on the other hand, presents an example in which the production function is $f(x, K)$ but where the externality from K is negative: he shows that a continuum of equilibria converging to a stationary state exists at appropriate parameter values. In the paper by Spear a third type of external effect is introduced: the production function is written as $f(x, K')$, where K' is tomorrow's aggregate capital stock which is assumed to have a positive effect on today's productivity. In this case the author derives a set of sufficient conditions under which stationary sunspot equilibria exist in a neighborhood of a stationary state.

This paper contains two more sections and the conclusions. The next one briefly summarizes the situation in the case of bounded accumulation paths, whereas Section 3 will discuss more extensively the models of perpetual growth.

2. BOUNDED GROWTH

We use this section to introduce the formal models and to provide a brief review of the bounded case. As we mentioned in the introduction the fact that indeterminacy cannot arise around a stationary state of a one-sector model with positive externalities, seems to be already a kind of "folk theorem." Therefore we avoid dwelling with it for too long, and concentrate instead on a simple example showing how easily indeterminacy arises in the bounded two-sector model. For a more extended discussion of these issues, as well as for the proofs of the statements reported here, the reader should consult the original working paper version of this article (Boldrin-Rustichini (1991)).

2.1. *The One-Sector Model*

The economy is composed of a continuum of identical agents indexed by $i \in [0,1]$. There is only one good which is used both as consumption and capital input. Each consumer i is infinitely lived and owns a firm and an initial stock of capital x_0^i. Given a sequence $\{k_t\}_{t=0}^{\infty}$ of aggregate capital stocks he chooses the consumption stream $\{c_t^i\}_{t=0}^{\infty}$ and the capital stocks' sequence $\{x_t^i\}_{t=0}^{\infty}$ that maximize his total discounted utility.

Each consumer owns a firm, with production function $G(x^i, k, l)$ depending on the private amount of capital stock x^i, the aggregate capital stock $k = \int_0^1 x^i di$, and labor l. The latter is inelastically supplied by the consumers and will be normalized to one. Except for the external factor, k, the production function G is standard. Denote with $0 \leqslant \mu \leqslant 1$ the capital depreciation rate. We define f: $\Re_+^2 \to \Re_+$ as $f(x^i, k) = G(x^i, k, 1) + (1-\mu)x^i$. For the purposes of this section the aggregate production function $F(x) = f(x, x)$ is also restricted to impede persistent growth.

ASSUMPTION 2.1: *The utility function* u: $\Re_+ \to \Re$ *is* C^2, *increasing, and strictly concave. The discount factor* δ *is in* $(0,1)$.

ASSUMPTION 2.2: G: $\Re_+^3 \to \Re_+$ *is of class* C^2. *For any given* $k \geqslant 0$ *it exhibits the following properties*:

(i) $G(\lambda x^i, k, \lambda l) = \lambda G(x^i, k, l)$, $\forall \lambda \geqslant 0$;

(ii) $G(\cdot, k, \cdot)$ *is increasing and concave*;

(iii) $G_{11}(\cdot, k, l) < 0$ *for all* l *and* $k > 0$.

ASSUMPTION 2.3: *The production function* $F(x) = f(x,x)$ *has the properties*:

(i) *There exists an* $\bar{x} > 0$ *such that* $F(x) > x$ *for* $0 < x < \bar{x}$ *and* $F(x) < x$ *for* $x > \bar{x}$.

(ii) *The partial derivative* f_1 *satisfies*: $f_1(\bar{x}, \bar{x}) < 1$ *and* $\lim_{x \to 0} f_1(x,x) > 1/\delta$.

Without loss of generality, we can assume for the remainder of this section that $x_0 \in [0, \bar{x}]$. Equilibria will then be sequences $\{x_t\}_{t=0}^{\infty}$ that, given a sequence $\{k_t\}_{t=0}^{\infty}$, solve the "parametric" programming problem:

$$\text{(P)} \qquad \max \left\{ \sum_{t=0}^{\infty} u(f(x_t, k_t) - x_{t+1})\delta^t \right\} \qquad \text{subject to}$$

$$0 \leqslant x_{t+1} \leqslant f(x_t, k_t)$$

together with the "fixed point problem" $x_t(\{k_t\}_{t=0}^{\infty}) = k_t$ for all t. In other words a sequence $\{x_t\}_{t=0}^{\infty}$ is an equilibrium for our economy if and only if it satisfies

$$\text{(EE)} \qquad u'(f(x_t, x_t) - x_{t+1}) = \delta u'(f(x_{t+1}, x_{t+1}) - x_{t+2}) f_1(x_{t+1}, x_{t+1}),$$

and

$$\text{(TC)} \qquad \lim_{t \to \infty} \delta^t x_t u'(f(x_t, x_t) - x_{t+1}) f_1(x_t, x_t) = 0.$$

The reader is invited to consult Kehoe-Levine-Romer (1991) for additional details. Before proceeding with our analysis we need to make precise our notion of indeterminacy. Intuitively we say that an equilibrium is indeterminate when there exists a whole interval of equilibrium paths starting off from its same initial condition. This, indeed, is the only way in which local uniqueness may fail to exist for an economy such as the one we study: once two initial conditions (say x_0 and x_1) are given, the dynamical system (EE) uniquely defines the rest of the equilibrium trajectory.

Definition 2.1: Let $\{x_t\}_{t=0}^{\infty}$ denote an equilibrium for an economy with initial condition $x_0 = k_0$. We say that it is an *indeterminate equilibrium* if for every $\epsilon > 0$ there exists another sequence $\{y_t\}_{t=0}^{\infty}$, with $0 < |y_1 - x_1| < \epsilon$ and $y_0 = x_0 = k_0$, which is also an equilibrium.

We have not yet specified the sign of the external effect. Kehoe (1991) shows that indeterminacy arises when negative external effects are present. Here we stress that when the externality is positive, equilibria converging to a stationary state are locally unique. Furthermore there exists a simple restriction on the technology which assures monotone convergence to a unique stationary state.

Theorem 2.1: *Assume $f_2(x, x) \geqslant 0$ holds. Then under Assumptions* 2.1, 2.2, *and* 2.3:

(i) *all equilibria converging to a stationary solution of* (*EE*) *are locally unique*;

(ii) *when the private return on capital $f_1(x, x)$ is a nonincreasing function of the capital stock, all interior equilibria are monotone increasing and unique. Moreover, there exists a unique value $x^* \in (0, \bar{x})$ such that if $x_0 \leqslant x^*$ then $\{x_t\}_{t=0}^{\infty}$ satisfies $x_t \leqslant x_{t+1} \leqslant x^*$ and if $x_0 \geqslant x^*$ then $\{x_t\}_{t=0}^{\infty}$ satisfies $x^* \leqslant x_{t+1} \leqslant x_t$ for every t.*

PROOF: The first statement can be verified by linearizing the (EE) around a steady state to verify that at least one of the eigenvalues will always be larger than one (in modulus) as long as the externality is positive. To prove statement (ii) we need only to show that all equilibria are monotone. We will articulate the proof in a lemmata.

LEMMA 1: *If $x_t \leqslant x^*$, then $c_t \geqslant c_{t-1}$ and if $x_t \geqslant x^*$ then $c_t \leqslant c_{t-1}$ (strict inequality in x implies strict inequality in c).*

PROOF: If $x_t \leqslant x^*$, $\delta f_1(x_t, x_t) \geqslant 1$ will hold, which implies $u'(c_{t-1})/u'(c_t) = \delta f_1(x_t, x_t) \geqslant 1$ and so $c_t \geqslant c_{t-1}$ because u is concave. Similarly when $x_t \geqslant x^*$.

LEMMA 2: *If $x_t \leqslant x^*$, then $x_{t+1} \geqslant x_t$.*

PROOF: Lemma 1 already implies $c_t \geqslant c_{t-1}$. Assume that $x_{t+1} < x_t$. Then (EE) implies

$$(*) \qquad \frac{u'(c_{t-1})}{u'(c_t)} = \frac{u'(c_t) f_1(x_t, x_t)}{u'(c_{t+1}) f_1(x_{t+1}, x_{t+1})} \geqslant 1.$$

We will show that a contradiction with (*) arises. To do this, notice first that $x_{t+1} < x_t$ implies $c_{t+1} < c_t$. In fact, if $c_{t+1} \geqslant c_t$ and $x_{t+1} < x_t \leqslant x^*$, then $x_{t+2} = F(x_{t+1}) - c_{t+1} < F(x_t) - c_t = x_{t+1}$ and so $x_{t+2} < x^*$, which implies (by Lemma 1) that $c_{t+2} \geqslant c_{t+1}$. This in turn gives $x_{t+3} = F(x_{t+2}) < F(x_{t+1}) - c_{t+1} = x_{t+2}$. By iteration the sequence $\{x_{t+i}\}_{i=0}^{\infty}$ satisfies $x_{t+i} < x_{t+i-1} \leqslant x^*$ for all $i \geqslant 1$ and the sequence $\{c_{t+i}\}_{i=0}^{\infty}$ satisfies $c_{t+i} \geqslant c_{t+i-1} > 0$ for all $i \geqslant 1$. Let $\underline{x} < x^* = \lim_{i\to\infty} x_{t+i}$ and $\bar{c} = \lim_{i\to\infty} c_{t+i}$. Then $\bar{c} > 0$, and $\bar{c}$ is finite because $\underline{x} < x^*$ implies $f(\underline{x}, \underline{x})$ is bounded. Hence, $u'(\bar{c}) \in (0, \infty)$ and $\delta f_1(\underline{x}, \underline{x}) = 1$ has to hold, which contradicts $\underline{x} < x^*$. So $x_{t+1} < x_t$ implies $c_{t+1} < c_t$.

Now recall that f_1 is nonincreasing and u' is decreasing; then $x_{t+1} < x_t$ implies

$$u'(c_t) f_1(x_t, x_t) \leqslant u'(c_t) f_1(x_{t+1}, x_{t+1}) < u'(c_{t+1}) f_1(x_{t+1}, x_{t+1}),$$

which contradicts (*). Therefore, $x_t \leqslant x^*$ implies $x_{t+1} \geqslant x_t$.

LEMMA 3: *If $x_t \leqslant x^*$, then $x_t \leqslant x_{t+1} \leqslant x^*$.*

PROOF: Only the part $x_{t+1} \leqslant x^*$ needs to be proved. Again, pretend $x_{t+1} > x^*$. Then (by Lemma 1) $c_{t+1} < c_t$ will hold and $x_{t+2} = F(x_{t+1}) - c_{t+1} > F(x_t) - c_t = x_{t+1}$ and, as in Lemma 2, iterations will give two sequences, $\{x_{t+i}, c_{t+i}\}_{i=0}^{\infty}$ with $x_{t+i+1} > x_{t+i} > x^*$ and $c_{t+i} < c_{t+i-1}$. Once again set $\lim_{i\to\infty} x_{t+i} = \bar{x} > x^*$ and $\lim_{i\to\infty} c_{t+i} = \underline{c}$. If $\underline{c} > 0$, then $u'(\underline{c})$ is finite and $\delta f_1(\bar{x}, \bar{x}) = 1$ has to hold, which contradicts $\bar{x} > x^*$. If $\underline{c} = 0$ and $u'(\underline{c})$ is not finite then, for i large enough, $f_1(x_{t+i}, x_{t+i}) \leqslant \gamma < 1$ must hold. Hence, $u'(c_{t+i+1}) = [\delta f_1(x_{t+i+1}, x_{t+i+1})]^{-1} u'(c_{t+i}) \geqslant (\delta\gamma)^{-1} u'(c_{t+i})$, which implies $u'(c_{t+i}) \geqslant$

$(\delta\gamma)^{-t}\underline{u}$ (for some constant $\underline{u}$ and t large). The latter gives

$$\lim_{t\to\infty} x_t f_1(x_t, x_t)\delta^t u'(c_t) \geq \lim_{t\to\infty} x_t f_1(x_t, x_t)(\gamma)^{-t}\underline{u} = +\infty.$$

This contradicts (TC) and proves the Lemma.

LEMMA 4: *If* $x_t \geq x^*$, *then* $x_t \geq x_{t+1} \geq x^*$.

PROOF: One needs only to replicate the proofs to Lemmata 1–3, with the appropriate changes in the inequalities. Now Lemmata 3 and 4, together with our initial observation about the eigenvalues of the linearized Euler Equation are equivalent to the second statement of Theorem 2.1. *Q.E.D.*

The restriction on the behavior of the private return on capital is necessary to deliver the result. One can in fact show that cycles emerge when the positive external effect is strong enough to make the private return on investments $f_1(x, x)$ an increasing function of the capital stock. One such example can be found in Boldrin-Rustichini (1991).

2.2. *The Two-Sector Model*

In this subsection we make the assumption that consumption and capital are different commodities produced by different combinations of labor and capital. We will show that this is enough to generate robust examples of indeterminate equilibria.

We retain here the market and demographic structures used before. On the production side there are two sectors; within each sector firms are identical and each consumer owns the same initial amount k_0 of capital stock and supplies a fixed unitary amount of labor in each period. Capital can be freely shifted from one sector to the other at the beginning of each production period. There is an external effect in production, which may affect either one or both production processes. Such external effect comes from the aggregate stock of capital and can be given any of the many interpretations found in the recent literature.

Let the production function of a typical firm in either sector be denoted as $F^i(x_t^i, l_t^i, k_t)$, with $i = 1$ for consumption and $i = 2$ for investment. We assume that, given the aggregate stock of capital k_t, both $F^i(\cdot, \cdot, k)$'s satisfy Assumption 2.2. Assuming that markets are fully competitive in every other respect one can define the Production Possibility Frontier (PPF) faced by a representative individual as

$$T(x_t, x_{t+1}, k_t) = \max_{x_t^1, l_t^1} F^1(x_t^1, l_t^1, k_t) \quad \text{subject to}$$

$$x_{t+1} \leq F^2(x_t^2, l_t^2, k_t) + (1-\mu)x_t,$$

$$x_t^1 + x_t^2 \leq x_t,$$

$$l_t^1 + l_t^2 \leq 1,$$

where x_t denotes the private and k_t the aggregate stock of capital. The parameter $\mu \in [0,1]$ is the capital depreciation factor and one is the total amount of labor available to an individual in each period.

Now denote with $u(c)$ the representative individual utility function and with $V(x, x', k)$ the composition $u(T(x, x', k))$. Then, as in the one sector model above, interior equilibria can be characterized by means of a variational equation (EE) and a transversality condition (TC). In the notation just introduced they are

$$\text{(EE)} \qquad V_2(x_t, x_{t+1}, x_t) + \delta V_1(x_{t+1}, x_{t+2}, x_{t+1}) = 0$$

and

$$\text{(TC)} \qquad \lim_{t \to \infty} \delta^t x_t V_1(x_t, x_{t+1}, x_t) = 0,$$

respectively. Linearization of (EE) around a steady state x^* gives the characteristic equation

$$\text{(2.1)} \qquad \lambda^2 + \lambda \left\{ \frac{V_{22}}{\delta V_{21}} + \frac{V_{11} + V_{13}}{V_{21}} \right\} + \left\{ \frac{1}{\delta} + \frac{V_{23}}{\delta V_{12}} \right\} = 0$$

where it should be understood that the functions V_{ij}, $i, j = 1, 2, 3$ are evaluated at the steady state. Our contention is that there exists an admissible set of parameter values at which both roots of (2.1) are inside the unit circle. In such circumstances equilibria are indeterminate, as x_0 near x^* implies that for all x_1 in an ϵ-ball around x_0 the path $(x_0, x_1, \dots)$ is an equilibrium converging to x^*. The necessary and sufficient conditions for both roots of a quadratic equation of the type $\lambda^2 + a_1 \lambda + a_2 = 0$ to be inside the unit circle are

$$(1 - a_2) > 0; \qquad (1 + a_1 + a_2) > 0; \qquad (1 - a_1 + a_2) > 0.$$

For equation (2.1) they translate into:

$$\text{(2.2)} \qquad \begin{cases} \dfrac{1}{\delta} + \dfrac{V_{23}}{\delta V_{12}} < 1; \\[2ex] 1 + \dfrac{1}{\delta} + \dfrac{V_{23} + V_{22}}{\delta V_{21}} + \dfrac{V_{11} + V_{13}}{V_{21}} > 0; \\[2ex] 1 + \dfrac{1}{\delta} + \dfrac{V_{23} - V_{22}}{\delta V_{21}} - \dfrac{V_{11} + V_{13}}{\delta V_{21}} > 0. \end{cases}$$

A careful examination of (2.2) shows that, contrary to the one-sector model, there exists economic conditions under which the three inequalities are simultaneously satisfied. In fact if V_{12} and V_{23} have opposite signs, the first condition can be obtained. Of the other two, only one is really binding; notice also that whatever sign a_1 may have, its magnitude can be made quite small by forcing

V_{11} and V_{13} to cancel each other. More precisely our economy has to display these three properties:

(i) a steady state value such that the consumption sector has a higher capital-labor ratio than the investment sector ($T_{12} < 0$) and a relatively inelastic marginal utility of consumption ($V_{12} = u'T_{12} + u''T_2T_1 < 0$);

(ii) a positive externality that also reduces the cost (in utils) of producing additional capital stock ($V_{23} = u'T_{13} + u''T_2T_3 > 0$);

(iii) an external effect that increases the marginal value of the current stock of capital together with a moderately concave utility function ($V_{13} = u'T_{13} + u''T_1T_3 > 0$).

None of these conditions appear economically unreasonable and they are not difficult to formalize. The example we provide next is just the simplest we could come up with. Other, more "realistic" ones, can be derived from more elaborated and better specified two-sector economies.

EXAMPLE 2.1: Begin by choosing a linear utility function $u(c) = c$, so that $V(x, x', k) = T(x, x', k)$. The same results would carry through with, say, a CES utility function, only the algebra would be messier. The output of the consumption good is given by $c = (l^1)^{\alpha}(x^1)^{1-\alpha}$ and output of the investment good is given by $y = \min\{l^2, x^2/\gamma\}$, with $\alpha, \gamma \in (0, 1)$. The aggregate stock of capital k also has the effect of increasing the efficiency level of the otherwise exogenous unitary labor supply. In other words the external effect is assumed to be observationally equivalent to labor-augmenting technological progress. Denoting with l_t the total number of efficiency units of labor at time t, we represent the externality as $l_t = k_t^{\eta}$. The allocational constraint is then $l_t^1 + l_t^2 \leqslant l_t$, for each t. To simplify further we will also assume instantaneous depreciation. The PPF for the representative agent is then given by

$$T(x, x', k) = (k^{\eta} - x')^{\alpha}(x - \gamma x')^{1-\alpha}.$$

Equilibria are those sequences $\{x_t\}_{t=0}^{\infty}$ that, given a sequence $\{k_t\}_{t=0}^{\infty}$, solve the following parametric programming problem:

$$(2.3) \qquad \max \sum_{t=0}^{\infty} \delta^t (k_t^{\eta} - x_{t+1})^{\alpha}(x_t - \gamma x_{t+1})^{1-\alpha} \quad \text{subject to}$$

$$0 \leqslant x_{t+1} \leqslant \min\left\{k_t^{\eta}, \frac{x_t}{\gamma}\right\}$$

and that also satisfy $x_t = k_t$ for all $t = 0, 1, 2, \ldots$.

The unique interior steady state solution to (2.3) is computed by solving the equation $T_2(x^*, x^*, x^*) + \delta T_1(x^*, x^*, x^*) = 0$, which gives

$$x^* = \left\{\frac{(\delta - \gamma)(1 - \alpha)}{(\delta - \gamma)(1 - \alpha) + \alpha(1 - \gamma)}\right\}^{1/1-\eta}.$$

Some tedious but nevertheless straightforward algebra will now prove the following theorem.

THEOREM 2.2: *There exists an open set of values in the parameter space* $(\alpha, \delta, \eta, \gamma)$, *such that the equilibria of the growth model* (2.3) *are indeterminate.*

PROOF: In light of the previous discussion it suffices to show the existence of some combinations of parameters at which the inequalities (2.3) are satisfied. The constants a_1 and a_2 can be computed as

$$a_1 = \frac{z^{-1} - \gamma}{\delta} + \frac{1 - \eta(x^*)^{\eta-1} z^{-1}}{z^{-1} - \gamma},$$

$$a_2 = \frac{1 - \eta(x^*)^{\eta-1} z^{-1}}{\delta},$$

where

$$z = \frac{(x^*)^{\eta-1} - 1}{1 - \gamma}.$$

It is then a simple numerical matter to verify that, for example, in a neighborhood of the parameter values $\alpha = .5$, $\delta = .5$, $\eta = .5$, and $\gamma = .2$, the inequalities (2.2) are all satisfied. The statement then follows from the continuity of the functions in (2.2). *Q.E.D.*

3. UNBOUNDED GROWTH

In this section we show that parallel conclusions hold also in the presence of persistent growth. More precisely we will prove that in the one-sector model, under reasonable hypotheses, equilibria are unique in the following sense: given a "large enough" initial condition x_0, there exists at most one sequence $\{x_t\}_{t=0}^{\infty}$ satisfying (EE) and (TC). Also, the asymptotic growth rate is unique: all equilibrium sequences must eventually grow at the same speed. Models in which the asymptotic growth rate is not bounded and in which the stock of capital grows infinitely big infinitely fast are not captured by our analysis. For the case of two sectors we show, by means of another example, that indeterminate growth paths cannot be ruled out even under very restrictive conditions.

3.1. *The One-Sector Model*

Assumptions 2.1 and 2.2 are maintained and only positive external effects will be considered. Our argument will proceed along these steps: first we show that (under only the extra assumptions required to guarantee unbounded accumulation) equilibrium orbits are locally unstable, thereby preventing nearby equilibria from merging into each other asymptotically. Then we introduce a set of additional assumptions about the curvature of the utility and production functions. This allows us to prove that when a constant growth rate exists it is uniquely and dynamically unstable, thereby implying the existence of at most one equilibrium path growing asymptotically at a constant rate.

We begin by assuming that the external effect is positive and that unbounded growth at a bounded rate is feasible:

ASSUMPTION 3.1: *The aggregate production function* $F(x)=f(x,x)$ *satisfies*:

- $f_2(x,x)\geqslant 0$;
- $\liminf_{x\to+\infty}[F(x)-x]>0$;
- $\liminf_{x\to+\infty}f_1(x,x)>\delta^{-1}$;
- $\lim_{x\to+\infty}F(x)/x=L<+\infty$.

One can verify that the last three parts of Assumption 3.1 together with strict concavity of the utility function imply that equilibrium consumption sequences are monotone increasing. This, together with feasibility considerations also implies that the capital stock sequence is monotone increasing along an equilibrium trajectory. Notice also that the third part of Assumption 3.1 effectively bounds the capital growth rate by L and, for x large, it implies $F(x)=Lx+g(x)$ with $\lim_{x\to\infty}g(x)/x=0$.

To build up some intuition on why orbits satisfying (EE) cannot converge to each other, pick one of them $\{x_t\}_{t=0}^{\infty}$ and compute the linear approximation to (EE) in a neighborhood of such an orbit. The associated Jacobian matrix is time dependent and with some algebra one can check that its two real roots, at any regular point of the trajectory $\{x_t\}_{t=0}^{\infty}$, are given by small perturbations of the following expressions:

$$\lambda_t^1=\frac{u''(c_t)}{u'(c_t)}\frac{u'(c_{t+1})}{u''(c_{t+1})};\qquad \lambda_t^2=F'(x_t).$$

The latter are exact when x_t and c_t are large enough. By Assumption 3.1 and the hypothesis that the external effect is positive, $\lambda_t^2>1$ for all t. A simple application of well known results from dynamical systems (see, e.g., Irwin (1980, page 114)) implies that trajectories are locally unstable at least along one direction.

To derive a formal proof of our claim we need some additional notation and a couple of extra hypotheses on the asymptotic behavior of the utility and production functions. In (EE) write $x_t=x$, $x_{t+1}=\lambda_t x$, $x_{t+2}=\lambda_{t+1}\lambda_t x$, to obtain a parameterized implicit function

$$\text{(EE)}\qquad \psi(x,\lambda_t,\lambda_{t+1})=-u'(F(x)-\lambda_t x)+\delta u'(F(\lambda_t x)-\lambda_{t+1}\lambda_t x)f_1(\lambda_t x,\lambda_t x)=0.$$

For all finite values of x, strict concavity of u guarantees the existence of a continuous function $\theta_x\colon\mathfrak{R}_+\to\mathfrak{R}$, mapping the current growth rate of capital λ_t

into λ_{t+1}, the growth rate in the subsequent period, and satisfying

$$(3.1)\qquad \psi(x,\lambda,\theta_x(\lambda))=0.$$

In general the map θ_x depends on the value of x, the current stock of capital, and the latter changes in each period. We are therefore facing a sequence of such maps θ_x. On the other hand, we are interested only in the behavior of θ_x at "large" values of x. One then needs to study the properties of the function

$$(3.2)\qquad \theta_\infty(\lambda)=\lim_{x\to\infty}\theta_x(\lambda),$$

over the interval $[l, L]$ for some $l>0$.

To carry this out we need the following assumption.

ASSUMPTION 3.2: *The private rate of return is eventually decreasing in the capital stock, i.e. there exists an* $\bar{x}$ *such that for all* $x \geqslant \bar{x}$, $\pi(x)=f_1(x,x)$ *is a nonincreasing function.*

ASSUMPTION 3.3: *Given* c *and* $c' \geqslant 0$ *define*

$$\sigma(c,c')=\frac{u''(c)\cdot u'(c')}{u''(c')\cdot u'(c)}.$$

Then

$$c'\geqslant c \textit{ implies } \sigma(c,c')\leqslant \frac{c'}{c}.$$

ASSUMPTION 3.4: *Given two pairs* (c,c') *and* $(\bar{c},\bar{c}')\in\Re^2_+$, *if* $u'(c)/u'(c')>u'(\bar{c})/u'(\bar{c}')$, *then* $\sigma(c,c')\geqslant\sigma(\bar{c},\bar{c}')$.

Assumption 3.2 prevents the private rate of return from continuously oscillating between a lower and an upper bound. This condition is necessary for the existence of a constant growth rate equilibrium. Along such equilibrium the stock of capital and the level of consumption must be growing at the same constant rate: this follows from Assumption 3.1 on the asymptotic linearity of the production function. Assumption 3.3 requires the utility function to display a nonincreasing elasticity of substitution in consumption. Uniqueness of the constant growth rate is mostly a consequence of this condition. Assumption 3.4 is a technical regularity restriction, satisfied by most of the commonly adopted utility functions. Its purpose is to guarantee that the asymptotic function θ_∞ is well behaved.

LEMMA 3.1: *Let* $\{x_t\}_{t=0}^{\infty}$ *be an equilibrium sequence with initial condition* $x_0 \geqslant \bar{x}$. *Then for some* $0 < l < 1$, *the sequence of functions* θ_{x_t}: $[l, L] \to [0, L]$ *defined in* (3.1) *is a monotone increasing sequence of continuous and monotone increasing functions. Furthermore the limit function* $\theta_\infty(\lambda)$ *defined in* (3.2) *exists and has the following properties*:

(i) *It is Lipschitz continuous, monotone increasing and concave over the interval* $[l, L]$.

(ii) *There exist at most two fixed points of* θ_∞; *call them* $1 \leqslant \lambda_1 < \lambda_2 \leqslant L$.

(iii) *The smallest fixed point is dynamically unstable while the other is stable.*

PROOF: The properties of θ_x for x finite can be derived by repeatedly applying the implicit function theorem to equation (3.1) and noticing that $x_t > \bar{x}$ must hold for t sufficiently large. To derive the properties of θ_∞ compute the slope of $\theta_x(\lambda)$ at two different values $\lambda < \tilde{\lambda}$, for given x. The difference $\theta_x'(\lambda) - \theta_x'(\tilde{\lambda})$ reduces to

$$\left[\sigma(c, c') - \sigma(\tilde{c}, \tilde{c}')\right] + \left[\theta_x(\tilde{\lambda}) - \theta_x(\lambda)\right] + r(x)$$

where the pairs (c, c') and $(\tilde{c}, \tilde{c}')$ are associated respectively with the trajectory departing from λx and the trajectory departing from $\tilde{\lambda} x$, and $r(x)$ is a term which, because of Assumption 3.2, becomes negligible when x is large. As θ_x is increasing the second term is positive and the first is made nonnegative by Assumption 3.4. This implies that $\theta_x(\lambda)$ is concave for x large enough. The sequence $\theta_{x_t}(\lambda)$ is uniformly bounded for all $\lambda \in [l, L]$ so it will converge pointwise. A standard theorem in convex analysis (see Rockafellar (1970, page 90)) guarantees the convergence is uniform and the limit function θ_∞ is therefore continuous and monotone increasing. It is also concave and therefore Lipschitzian. The existence of, at most, two fixed points and their instability/stability then follow. With "dynamically stable/unstable" we mean that the slope of θ_∞ measured at λ_2 is less than one whereas it is larger than one at λ_1.

Q.E.D.

It is useful and of some interest to compare the properties of the functions θ_x and θ_∞, with the corresponding functions for the optimal growth problem, defined as the maximization in (P) taken over the sequences $\{x_t, k_t\}_{t=0}^{\infty}$, such that $x_t = k_t$ for all t. We denote by $\hat{\theta}_x$ and $\hat{\theta}_\infty$ these functions. Assume the optimal growth problem is well defined (i.e. concave). Then its Euler Equation is going to be similar to (EE), but with $f_1(x_{t+1}, x_{t+1})$ replaced by $f_1(x_{t+1}, x_{t+1}) + f_2(x_{t+1}, x_{t+1})$. Assumption 3.1 then implies that for all the values of λ satisfying $F(\lambda x) - \hat{\theta}_\infty(\lambda)\lambda x \geqslant 0$, the inequalities

$$\hat{\theta}_x(\lambda) \leqslant \theta_x(\lambda), \qquad \hat{\theta}_\infty(\lambda) \leqslant \theta_\infty(\lambda)$$

will hold. Furthermore properties (i), (ii), and (iii) of Lemma 3.1 extend to the functions $\hat{\theta}_x$ and $\hat{\theta}_\infty$.

THEOREM 3.1: *Let Assumptions* 2.1, 2.2 *and* 3.1–3.4 *be satisfied.*

(*a*) *Given any initial condition* $x_0 \geqslant \bar{x}$ *if an equilibrium exists it is unique.*

(*b*) *Along such equilibrium the growth rate of the capital stock* $\lambda_t = x_{t+1}/x_t$ *satisfies* $\lambda_{t+1} \leqslant \lambda_t$ *and converges to a constant growth rate* $\lambda^* = \lim_{x \to \infty} \lambda_1(x)$, *where the latter is the smaller fixed point of the map* $\theta_x(\lambda)$.

(*c*) *The equilibrium growth rate decreases as the capital stock increases.*

PROOF: Take an arbitrary initial condition $x \geqslant \bar{x}$ and assume $\{x_t\}_{t=1}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ are two distinct equilibria departing from it. Then $\lambda_0^x = x_1/x_0 \neq y_1/y_0 = \lambda_0^y$. To fix ideas set $\lambda_0^y > \lambda_0^x$. Then the sequence $\{y_t\}_{t=1}^{\infty}$ will dominate $\{x_t\}_{t=1}^{\infty}$ coordinatewise and it follows from the previous Lemma that $\lambda_t^y > \lambda_t^x$ must hold for all $t \geqslant 0$. Inspection of the first order conditions then yields $c_t^x > c_t^y$ for all $t \geqslant 0$. Now consider the consumer optimization problem when the given aggregate sequence is $\{y_t\}_{t=0}^{\infty}$: he can (for example) pick $\{x_t\}_{t=0}^{\infty}$ (which is feasible) and obtain in each period a consumption level $c_t = f(x_t, y_t) - x_{t+1} > c_t^x > c_t^y$. The sequence $\{y_t\}_{t=1}^{\infty}$ therefore cannot be an equilibrium. This proves (a). To prove (b) notice that the equilibrium growth rate sequence must be decreasing because if it were increasing even only for one period it would have to be increasing forever. Under our assumptions this would yield a sequence $\{\lambda_t\}_{t=0}^{\infty}$ converging either to the highest fixed point of θ_∞ or to infinity, and therefore induce a consumption sequence which would be suboptimal. The equilibrium sequence of growth rates then must converge to λ^* "from below," i.e. along a trajectory such that each pair $(\lambda_t, \lambda_{t+1})$ belongs to that portion of the graph of θ_{x_t} which is below the diagonal. The latter proves (c). *Q.E.D.*

The reader should notice that we always assume an equilibrium exists: this is because existence depends on the fact that the chosen parameter values satisfy the transversality condition over and above the recursive equation (EE). We should also add that part (a) of the theorem could be proved directly by showing that all those paths that converge to the "high steady state" of θ_∞ violate the transversality condition: this can be accomplished by comparing their asymptotic behavior to that of paths driven by the "optimal" map $\hat{\theta}_\infty$ we mentioned before.

A few examples should facilitate intuition. In the first one Assumption 3.3 is satisfied as an equality for all consumption levels. The convergence to the asymptotic function θ_∞ is therefore instantaneous. In the second example, the same condition is satisfied as a strict inequality and the process of convergence is instead asymptotic. Finally the third example is meant to illustrate how a utility function which violates Assumption 3.3 would destroy our result.

EXAMPLE 3.1: Let $u(c) = c^{1-\gamma}/(1-\gamma)$, $f(x,k) = ax + bx^{\alpha}k^{1-\alpha}$ with $a, b > 0$, $\alpha \in (0,1)$. It is immediate to verify that when $\delta(a + \alpha b) > 1$ all of our assumptions hold. The asymptotic function θ_∞ in this case can be computed directly and is given by

$$\theta_\infty(\lambda) = L - [\delta(a + \alpha b)]^{1/\gamma} \frac{L - \lambda}{\lambda}. \tag{3.3}$$

The two asymptotic roots are therefore

$$\begin{cases} \lambda_1 = [\delta(a+\alpha b)]^{1/\gamma}, \\ \lambda_2 = L = a+b. \end{cases}$$

Here two different cases are still possible:

Case 1: $\lambda_1 \leqslant L$; then no equilibrium exists that satisfies our hypotheses, because both growth rates conflict with the transversality condition.

Case 2: $\lambda_1 < \lambda_2$; then there is a unique equilibrium growth path if the transversality condition is satisfied. The latter requires $\delta(a+\alpha b)^{1-\gamma} < 1$. In these circumstances it is easy to verify that the asymptotic map (3.3) is unstable at the fixed point λ_1.

EXAMPLE 3.2: Let the utility function be $u(c) = -\exp(-c)$ and take a general production function. The Euler Equation $\psi(x, \lambda_t, \lambda_{t+1}) = 0$ becomes

$$\exp[-(F(x)-\lambda_t x)] = \delta \exp[-(F(\lambda_t x) - \lambda_t \lambda_{t+1} x)] f_1(\lambda_t x, \lambda_t x), \tag{3.4}$$

which can be reduced to

$$F(x) - F(\lambda_t x) + \lambda_t \lambda_{t+1} x - \lambda_t x + k(x, \lambda_t) = 0 \tag{3.5}$$

where $k(x, \lambda_t) = \log(\delta f_1(\lambda_t x, \lambda_t x))$. Dividing both sides of (3.5) by x and rearranging we have

$$\theta_x(\lambda) = \frac{F(\lambda x)}{\lambda x} - \frac{F(x)}{\lambda x} + 1 - \frac{k(x,\lambda)}{\lambda x}$$

which satisfies all the properties listed in Lemma 3.1. Taking limits as $x \to \infty$ one finally obtains the asymptotic function θ_∞, which is

$$\theta_\infty(\lambda) = L - \frac{L}{\lambda} + 1.$$

The unique asymptotic equilibrium growth rate is therefore $\lambda^* = 1$ to which the economy converges as the stock of capital goes to infinity. Note that the asymptotic Euler Equation is not verified as an equality here, at least as long as $\delta\pi(x) > 1$ holds. The equilibrium sequence is one along which capital stock and consumption grow unbounded at a decreasing rate and become constant only "at infinity."

EXAMPLE 3.3: Again we need not restrict the production function to any particular form. Assume the marginal utility of consumption is given by $u'(c) = 1/\log(c+1)$. The latter does not satisfy Assumption 3.3.

By rearranging the Euler Equation for finite values of x one obtains

$$\theta_x(\lambda) = \frac{F(\lambda x)}{\lambda x} - \frac{[F(x)-\lambda x]^{\delta f_1(\lambda_x, \lambda x)}}{\lambda x}. \tag{3.6}$$

By inspection one will observe that the sequence of functions generated by (3.6) converges in the limit to a discontinuous function equal to $-\infty$ for $\lambda < L$ and to $+\infty$ for $\lambda > L$. The growth rate $\lambda = L$ is a fixed point of such a function but it is not an equilibrium for obvious reasons. Therefore there is no asymptotic equilibrium satisfying Theorem 3.1.

3.2. *The Two-Sector Model*

As mentioned in the introduction indeterminacy is also possible for the two-sector model in the presence of endogenous growth. Again we will be satisfied by making our point with a very simple, almost trivial, example.

To better illustrate the equilibrium behavior in the presence of externalities we begin this subsection with a brief analysis of the standard case. Once again there are two goods: a consumption good produced with a Cobb-Douglas technology $c = (x_1)^{\alpha}(l_1)^{1-\alpha}$, and an investment good produced with a linear one, $i = bx_2$. The aggregate capital stock x_t induces the constraint $x_t \geqslant x_{1t} + x_{2t}$, and evolves according to the law of motion $x_{t+1} = (1-\mu)x_t + i_t$. We introduce a few innocuous simplifications: the utility function is linear and the exogenous labor supply l is set equal to one in every period.

One can write the PPF as $T(x, x') = (\gamma x - ax')^{\alpha}$, with $\gamma = 1 + (1-\mu)/b > 1$, and $a = 1/b$. The Euler Equation associated to this simple optimization problem can be easily manipulated to yield a one dimensional map from current to future growth rates of the stock of capital:

$$\lambda_{t+1} = \tau(\lambda_t) \equiv \theta + (\delta\theta)^{1/1-\alpha} - \theta(\delta\theta)^{1/1-\alpha}\lambda_t^{-1} \tag{3.7}$$

where $\theta = b + (1-\mu) > 1$ is necessary to make persistent growth feasible. The function τ has two fixed points,

$$\lambda_1 = \theta, \quad \text{and} \quad \lambda_2 = (\delta\theta)^{1/1-\alpha}.$$

The first root, $\lambda_1 = \theta$, should be ruled out as a possible equilibrium with constant growth as consumption is forever zero along such an accumulation path. For the second root to be an equilibrium we need to verify that the transversality condition is satisfied. At λ_2, (TC) requires $\delta\theta^{\alpha} < 1$. The latter inequality also guarantees that $\lambda_2 < \lambda_1$ and that λ_2 is an unstable fixed point of τ.

As we should have expected, in an optimal growth model without any external effect if an equilibrium with persistent growth exists it is also determinate.

We shall now proceed to modify this model by appending an external effect to the production function of the consumption good. Set $c = k^{\eta}(x^1)^{\alpha}$. Then the PPF faced by a representative consumer-producer becomes

$$c_t = k_t^{\eta}(\gamma x_t - ax_{t+1})^{\alpha} \tag{3.8}$$

where, as usual, k_t denotes the aggregate capital stock which is treated parametrically by the representative agent. Given a $\{k_t\}_{t=0}^{\infty}$ equilibria are sequences $\{x_t\}_{t=0}^{\infty}$ solving

$$(3.9) \qquad \max \sum_{t=0}^{\infty} k_t^{\eta}(\gamma x_t - a x_{t+1})^{\alpha} \delta^t \quad \text{subject to}$$

$$0 \leqslant x_{t+1} \leqslant \theta x_t$$

and satisfying $x_t = k_t$ for all t.

As in the previous treatment of the one-sector model we restrict ourselves to the study of sequences with bounded growth rates. In this example it is always true that

$$\limsup_{t \to \infty} \frac{x_{t+1}}{x_t} \leqslant \theta.$$

The functional forms have been chosen to guarantee that the Euler Equation associated to (3.9) can be written in the form $\psi(x, \lambda_t, \lambda_{t+1}) = 0$ and that by simple manipulation a map $\tau(\lambda_t) = \lambda_{t+1}$ can be derived that satisfies $\psi(x, \lambda, \tau(\lambda))$ independently of x. The latter is

$$(3.10) \qquad \lambda_{t+1} = \tau(\lambda_t) \equiv \theta - (\delta\theta)^{1/1-\alpha} \lambda_t^{\beta}(\theta - \lambda_t)$$

where $\beta = (\alpha + \eta - 1)/(1 - \alpha)$. Given an initial condition $\lambda_0 > 0$ every uniformly bounded trajectory of the dynamical system τ is candidate to be an equilibrium. In order to be one it has to satisfy the appropriate transversality condition. Among the bounded trajectories a special role is played by the fixed points and the closed orbits of τ, and our analysis will concentrate on them. Nevertheless, as we will briefly point out later, there are other more complicated orbits of τ that also satisfy (3.10) and therefore are equilibria. Some of them can be chaotic.

Along a balanced growth path with constant growth rate equal to λ the transversality condition reads as

$$(3.11) \qquad \lim_{t \to \infty} \alpha\gamma\delta^t x_t^{\eta+1}(\gamma x_t - a x_{t+1})^{\alpha-1} = \lim_{t \to \infty} \text{const} \cdot (\delta\lambda^{\alpha+\eta})^t = 0.$$

To prove our claim we only need to show that there exists a fixed point of τ that satisfies (3.11) and is asymptotically stable for the dynamics $\lambda_{t+1} = \tau(\lambda_t)$. This is spelled out in our last theorem. Generally, though, indeterminacy can also arise in the following more complicated fashion: there exists a subset $\Lambda \subset [0, \theta]$, which is an attractor for $\lambda_{t+1} = \tau(\lambda_t)$ and which contains a more than countable number of points. As the analysis of this case would lead us astray, we prefer to bypass it here. We refer the reader to Boldrin and Persico (1993) for a more detailed study.

THEOREM 3.2: *In the model of growth with externalities described by the programming problem* (3.9), *equilibria are indeterminate when the following restrictions are satisfied*:

- $\alpha + \eta > 1$,
- $\delta\theta < 1 < \delta\theta^{\alpha+\eta}$,
- $\lambda_2 > \theta - 1/\beta$.

Then $\lambda_2 = (\delta\theta)^{1/(1-\alpha-\eta)}$ *is the only constant growth rate that satisfies the transversality condition. It is also asymptotically stable under iterations of* (3.10).

PROOF: Derivation of (3.10) from (EE) is a simple matter of algebra. Similarly it is straightforward to verify that when $\alpha + \eta = 1$ the function τ has only one fixed point equal to θ. When $\alpha + \eta \neq 1$, τ has the two fixed points $\lambda_1 = \theta$, $\lambda_2 = (\delta\theta)^{1/(1-\alpha-\eta)}$. The transversality condition reduces to $\delta\lambda_i^{\alpha+\eta} < 1$. The case $\alpha + \eta < 1$ is similar to the model without externality. It is easy to see that the root λ_2 is the unique equilibrium and that it is unstable.

The case $\alpha + \eta > 1$ requires a few extra computations. Here $\beta > 0$, so that $\tau(0) = \theta > 1$, $\tau(\theta) = \theta$, and $\tau'(\lambda) = (\delta\theta)^{1/(1-\alpha)}\lambda^{\beta}(1 - \beta(\theta - \lambda))$. This implies, in particular that $\tau'(\lambda_1) > 0$ whereas $\tau'(\lambda_2)$ may be of either sign. The condition $\delta\theta^{\alpha+\eta} > 1$ guarantees at once that $\lambda_1 > \lambda_2$, and that λ_2 satisfies the transversality condition. To check that λ_2 is stable one has only to notice that τ has a minimum at $\lambda^* = \theta - 1/\beta$ and that our last condition is equivalent to $\lambda^* < \lambda_2$.

Q.E.D.

The form of indeterminacy described in our theorem is the familiar one in which for a given initial condition x_0 there exists an open interval of values of x_1 that are all consistent with equilibrium. These distinct trajectories grow asymptotically at a common rate λ_2 but need to converge to each other, i.e. they typically grow "parallel" forever. It is difficult to say if the parameter values at which this phenomenon occurs may be considered "realistic" or otherwise, mainly because the model we are using is rather simplified. To get an idea of the range of values involved we provide a rough parameterization of our model. Choose a depreciation rate of about 10% and a capital/output ratio around 3.4 in the investment sector to obtain a value of θ equal to 1.2. With a relatively low discount factor, say $\delta = .80$, one needs $\alpha = .5$, $\eta = 1$ to bring λ_2 around the "credible" value of 1.08. Then, as can be easily verified, the stability condition is also satisfied and equilibria are indeed indeterminate. Everything clearly relies on the magnitude of the externalities, a matter about which very little empirical evidence is available.

The indeterminate and chaotic equilibria we mentioned above arise at about the same parameter values when $\lambda_2 < \theta - 1/\beta$. $\tau(\lambda)$ is then a nonmonotone mapping of the interval $[0, \theta]$ into itself for which both stationary states λ_1 and λ_2 are dynamically unstable.

One final comment on the interpretation to be given to the last theorem and to the case of "chaotic indeterminacy" we just outlined: According to this model two countries that start from the same initial stock and follow different equilibria from then on will display a *common average growth rate* only in the long run. Their capital stocks may therefore be persistently different (because different values of x_1 were chosen) and we may well observe their relative economic conditions becoming increasingly different. In other words models of the type discussed here may not only account for the fact that certain countries never catch-up with the leader, but also for the more disturbing phenomenon that countries which started out from almost similar conditions a century ago, have been growing very differently since then. In particular one can think of examples in which a small difference in the choice of x_1 (given a common x_0) will induce two diverging sequences of capital stocks, growing at a common rate only in the distant future.

4. CONCLUSIONS

We have studied the determinacy of competitive equilibrium in infinite horizon models of capital accumulation with productive externalities.

In the standard one-sector model we have proved that equilibria converging to a steady state are always locally unique and that unbounded equilibria converging to a stationary growth rate are also locally unique under reasonably mild conditions.

We have also addressed the problem of indeterminacy within the context of a two-sector growth model again in the presence of an aggregate externality. In this case indeterminacy of equilibrium seems to be always possible and indeed appears quite easily even in the simplest models. For very standard functional forms of the utility and production functions and for parameter values that appear altogether not unreasonable there exists a continuum of distinct equilibria departing from a common initial stock of capital and either converging to the same steady state or growing asymptotically at a common rate.

The practical implications of these results cannot be fully evaluated given the simplified models adopted here. Further research along these lines should clarify if the phenomenon we have pointed out is robust with regards to a number of empirically relevant perturbations of the stylized models we have studied. From the point of view of the theory of economic development an important extension is to models with more than one stock of capital (physical and human) and to models of technological change and/or industrialization. From the point of view of business cycle theory one would be curious as to what implications an endogenous labor supply and more realistic production functions would have on the model's predictions about the interplay between endogenous growth and endogenous oscillations. From a general perspective it seems that the study of multisector growth models with external effects is a promising avenue for the long overdue reconciliation between the theory of economic growth and the theory of the business cycle.

MEDS-KGSM, Northwestern University, 2001 Sheridan Rd., Evanston, IL 60208, U.S.A.

and

Dept. of Economics, New York University, 269 Mercer St., New York, NY 10003, U.S.A.

Manuscript received February, 1992; final revision received August, 1993.

REFERENCES

Benhabib, J., and R. Farmer (1993): "Indeterminacy and Growth," mimeo, New York University and UCLA, forthcoming *Journal of Economic Theory*.

Benhabib, J., and R. Perli (1993): "Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth," mimeo, New York University, forthcoming *Journal of Economic Theory*.

Boldrin, M. (1992): "Dynamic Externalities, Multiple Equilibria and Growth," *Journal of Economic Theory*, 58, 198–218.

Boldrin, M., and N. Persico (1993): "A Chaotic Map Arising in the Theory of Endogenous Economic Growth," mimeo, Northwestern University.

Boldrin, M., N. Kiyotaki, and R. Wright (1993): "A Dynamic Equilibrium Model of Search, Production and Exchange," forthcoming *Journal of Economic Dynamics and Control*.

Boldrin, M., and A. Rustichini (1991): "Indeterminacy of Equilibrium in Dynamic Models with Externalities," Discussion Paper 955, The Center for Mathematical Studies in Economics and Management Sciences, Northwestern University.

Caballe, J., and M. Santos (1991): "On Endogenous Growth with Physical and Human Capital," forthcoming *Journal of Political Economy*.

Chamley, C. (1992): "Externalities and Dynamics in Models of Learning or Doing," forthcoming, *International Economic Review*.

Diamond, P., and D. Fudenberg (1989): "Rational Expectations Business Cycles in Search Equilibrium," *Journal of Political Economy*, 97, 606–618.

Gali, J. (1993): "Monopolistic Competition, Business Cycles and the Composition of Aggregate Demand," mimeo, Columbia Univ., forthcoming *Journal of Economic Theory*.

Howitt, P., and R. P. McAfee (1988): "Stability of Equilibria with Externalities," *Quarterly Journal of Economics*, 103, 261–277.

Irwin, M. (1980): *Smooth Dynamical Systems*. New York: Academic Press.

Kehoe, T. (1991): "Computation and Multiplicity of Equilibria," in *Handbook of Mathematical Economics*, Volume IV, ed. by W. Hildenbrand and H. Sonnenschein. Amsterdam: North-Holland.

Kehoe, T., and D. Levine (1985): "Comparative Statics and Perfect Foresight in Infinite Horizon Economies," *Econometrica*, 53, 433–453.

Kehoe, T., D. Levine, and P. Romer (1991): "On Characterizing Equilibria of Economies with Externalities and Taxes as Solutions to Optimization Problems," *Economic Theory*, 2, 43–68.

Lucas, R. E. Jr. (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22, 3–42.

Matsuyama, K. (1991): "Increasing Returns, Industrialization and Indeterminacy of Equilibrium," *Quarterly Journal of Economics*, 106, 617–649.

Mortensen, D. (1991): "Unemployment Equilibrium Cycles," Discussion Paper 939, The Center for Mathematical Studies in Economics and Management Sciences, Northwestern University.

Rockafellar, T. (1970): *Convex Analysis*. Princeton, NJ: Princeton University Press.

Romer, P. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71–S102.

Spear, S. (1991): "Growth, Externalities and Sunspots," *Journal of Economic Theory*, 54, 215–223.

Xies, D. (1993): "Divergences in Economic Performances: Transitional Dynamics with Multiple Equilibria," forthcoming *Journal of Economic Theory*.