

CHAOTIC EQUILIBRIUM DYNAMICS IN ENDOGENOUS GROWTH MODELS

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Abstract

We study a class of two-sector endogenous growth models in the presence of a positive external effect. The class of models exhibits global indeterminacy of equilibria. The qualitative properties of a set of examples are analyzed by means of analytical and numerical methods. We also construct robust examples of both topological and ergodic chaos.

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1. Introduction

In the theory of endogenous growth, a number of interesting phenomena have been demonstrated by introducing positive externalities into a deterministic dynamic equilibrium model with infinitely-lived agents (Lucas (1988) and Romer (1986, 1990)). Of particular relevance to this study is the indeterminacy of equilibria, which implies the existence of a continuum of equilibrium paths departing from the same initial condition (Benhabib and Farmer (1994), Benhabib and Perli (1994), and Boldrin and Rustichini (1994)). In the existing literature, these results are examined primarily for equilibrium paths lying in a small neighborhood of a balanced growth path (*local indeterminacy*). As a result, the global properties of the equilibrium paths outside such a neighborhood have not yet been fully understood.

In the present study, we construct a simple, aggregate, endogenous growth model with a positive externality and develop a method of characterizing the global equilibrium dynamics for the case in which equilibrium paths do not, in general, converge to a balanced growth path. Using this method, we demonstrate that positive externalities can cause (i) an equilibrium to be indeterminate over a global range of initial growth rates for any given initial capital stock (*global indeterminacy*), (ii) the growth rate along a particular equilibrium path to fluctuate chaotically, and (iii) the average growth rate over a given span of time to depend upon the choice of an initial growth rate for a given economy. These results demonstrate that in models in which unbounded accumulation is possible and production externalities are important, trend and cycles in aggregate national output can be simultaneously generated by the same endogenous economic mechanism. In this system, one may observe global indeterminacy of equilibria and derive robust examples of both topological and ergodic chaos.

The language we have adopted should be clarified. We say that the model exhibits “indeterminacy of equilibria” if there exists a continuum of equilibria associated to the same initial condition, which in our case is the aggregate initial capital stock. In growth models, the property of indeterminacy is often proved only for equilibria departing from initial growth rates which are close to the balanced growth rate. Almost always, such equilibria continue to stay in a small neighborhood of the balanced growth path and, more often than not, converge to it. In these circumstances we say that the model exhibits “local” indeterminacy of equilibria near the balanced growth path. Consider instead the hypothetical case in which equilibria are indeterminate but do not converge to any balanced growth

path nor are required to start from growth rates that are close to the balanced growth rate. In particular, imagine the case in which, no matter what the initial stock of capital is, equilibria are indeterminate and each equilibrium path does not remain near the balanced growth path as time increases. Then we say that the model exhibits “global” indeterminacy of equilibria. In this paper we pursue the conditions under which this type of equilibria may exist.

We achieve this by showing that an Euler path, which satisfies (1) the transversality condition and (2) the summability of the objective function, is an equilibrium path. We focus on chaotic equilibria such that (3) average growth rates are positive. When equilibria are locally indeterminate, there are relatively simple inequality constraints against which the conditions (1) to (3) can be tested along the balanced growth path. However, when we treat the “global” indeterminacy of equilibria and admit the presence of chaotic solutions in an unbounded growth model, we do not necessarily have such simple inequality constraints. In this paper, we present a systematic method to overcome the analytical difficulties created by the simultaneous presence of chaotic solutions and unbounded state variables.

By this method, we may demonstrate that a continuum of equilibrium paths exist even if the initial growth rate is not chosen near the balanced growth rate.² At certain parameter values, those equilibrium paths do not converge to any balanced growth path but behave chaotically. This implies that, even if an equilibrium path starts arbitrarily near the balanced growth path, it does not stay trapped inside any neighborhood of the balanced growth path (other than the entire region in which the system is defined). An implication of this fact is that, for given fundamental characteristics and even in the long run, two economic systems may grow at different growth rates along different equilibrium paths which depend only upon the initial choice of a non-predetermined variable: the first period growth rate. This theoretical feature of our model stands in sharp contrast with the case of local indeterminacy where, asymptotically, systems always grow at the same (balanced) growth rate along all equilibria.

In reality growth rates oscillate, sometimes wildly, around average values that also change from one country to another. In the literature on real business cycles, these phenomena are explained by means of a stochastic process underlying the economy.³ In this paper we advance a different theoretical explanation. Namely, the growth rate of the capital stock in a growing economy may oscillate as a result of non-linearities that are present in a deterministic equilibrium system

²For treatments of indeterminacy in the stochastic framework, see Cass and Shell (1983).

³See, for example, Kydland and Prescott (1982) and Hansen (1985).

with a positive externality. Chaotic fluctuations are persistent and give rise to a non-trivial, invariant distribution of growth rates.

The remaining of the paper is organized as follows. Section 2 illustrates the model and gives a general characterization of equilibrium paths in the presence of externalities and unbounded state variables. Section 3 shows under which conditions cyclical and various kinds of chaotic paths can be obtained. Section 4 illustrates the behavior of the main economic aggregates along a chaotic growth path and discuss some special cases and extensions. Section 5 briefly concludes.

2. The Model

Our model is based on the two-sector model with externality that was developed in Boldrin and Rustichini (1994). There are two goods produced in two different sectors: the consumption good and the investment good. Let c_t and I_t be the amount of the consumption good and that of the investment good that are produced in period t . The consumption good is produced from both capital K_{1t} and labor L . The investment good is produced from capital K_{2t} alone. Let k_t be the amount of the capital good that is available at time t . Then

$$K_{1t} + K_{2t} = k_t. \quad (2.1)$$

The initial amount of capital input is given by $k_0 = \bar{k} > 0$. The economy is also endowed with a fixed amount of labor $L = 1$.

Externalities are at work in the production of the consumption good. Denote by e_t the source of this externality. The production function of the consumption good sector (sector 1) is

$$c_t = e_t^{\bar{\eta}} K_{1t}^{\bar{\alpha}} L^{1-\bar{\alpha}}, \quad (2.2)$$

where $0 < \bar{\alpha} < 1$ and $\bar{\eta} > 0$. The production function of the investment good sector (sector 2) is

$$I_t = bK_{2t}. \quad (2.3)$$

Let μ be the rate of depreciation of the capital good. Assume $0 < \mu \leq 1$. Then

$$k_{t+1} = I_t + (1 - \mu)k_t. \quad (2.4)$$

Let $\theta = b + 1 - \mu$, and assume $\theta > 1$, which is necessary for sustained growth to be feasible.

Let $u(c_t)$ be the utility function of the representative consumer. Denote by δ , $0 < \delta < 1$, the discount factor of future utilities. Then, the representative agent's

behavior is described by the following optimization problem. Given a sequence of externalities e_t , $t = 1, 2, \dots$,

$$\begin{aligned}
& \max_{\{c_t, k_t, K_{1t}, K_{2t}, I_t\}_{t \geq 0}} \sum_{t=0}^{\infty} \delta^t u(c_t) & (2.5) \\
& \text{s.t. } c_t = e_t^{\bar{\eta}} K_{1t}^{\bar{\alpha}} \\
& I_t = bK_{2t} \\
& K_{1t} + K_{2t} = k_t \\
& k_{t+1} = I_t + (1 - \mu)k_t \\
& k_0 = \bar{k} .
\end{aligned}$$

The level of the externality generated in period t is equal to the amount of the capital good employed in that period by the two sectors, *i.e.*,

$$e_t = k_t. \quad (2.6)$$

The general properties of this model have already been studied in Boldrin and Rustichini (1994). In particular, it was shown there that the model has a continuum of equilibria with unbounded growth for certain ranges of parameter values.

In this study, we will focus on the dynamic properties of this set of equilibria. To this end, we adopt the following utility function.

$$u(c) = c^{1-\sigma}, \quad 0 < \sigma < 1. \quad (2.7)$$

Since $c_t = e_t^{\bar{\eta}} [(\theta k_t - k_{t+1})/b]^{\bar{\alpha}}$, the above optimization problem, (2.5), gives rise to

$$\max_{\{k_t\}_{t \geq 0}} \sum_{t=0}^{\infty} \delta^t e_t^{\eta} (\theta k_t - k_{t+1})^{\alpha} \quad \text{s.t. } k_0 = \bar{k} \text{ and } (1 - \mu)k_t \leq k_{t+1} \leq \theta k_t, \quad (2.8)$$

where $\eta = (1 - \sigma)\bar{\eta}$ and $\alpha = (1 - \sigma)\bar{\alpha}$. We may treat a path of accumulation $\{k_t\}_{t \geq 0}$ as an *equilibrium path* if (i) it solves the optimization problem (2.8) and (ii) $e_t = k_t$ for $t = 0, 1, 2, \dots$.

We call a path $\{k_t\}_{t \geq 0}$ satisfying $k_0 = \bar{k}$ and $(1 - \mu)k_t \leq k_{t+1} \leq \theta k_t$ a *feasible path* (from $k_0 = \bar{k}$). We call $\{k_t\}_{t \geq 0}$ a *balanced growth path* if it is in equilibrium and $k_{t+1}/k_t = \lambda$ for $t = 0, 1, \dots$.

3. Equilibrium Dynamics

3.1. Characterization of Equilibria

In order to characterize the equilibrium in our model, we will first characterize the solution to the maximization problem (2.8).

Lemma 1. *Let $\{k_t\}_{t \geq 0}$ be a feasible path from $k_0 = \bar{k}$. Then, it solves the maximization problem (2.8) if the following three conditions are satisfied.*

$$\text{Euler Equation: } -e_t^\eta(\theta k_t - k_{t+1})^{\alpha-1} + \delta \theta e_{t+1}^\eta(\theta k_{t+1} - k_{t+2})^{\alpha-1} = 0; \quad (3.1)$$

$$\text{Transversality Condition: } \lim_{t \rightarrow \infty} \delta^t e_t^\eta(\theta k_t - k_{t+1})^{\alpha-1} k_{t+1} = 0; \quad (3.2)$$

$$\text{Summability Condition: } \sum_{t=0}^{\infty} \delta^t e_t^\eta(\theta k_t - k_{t+1})^\alpha < \infty. \quad (3.3)$$

Proof: We will demonstrate that if $\{k_t\}_{t \geq 0}$ satisfies (3.1), (3.2) and (3.3), then it solves the optimization problem (2.8). Define

$$p_{t+1} = \alpha \delta^t e_t^\eta(\theta k_t - k_{t+1})^{\alpha-1}. \quad (3.4)$$

Since, by (3.1), $p_t = \alpha \delta^t \theta e_t^\eta(\theta k_t - k_{t+1})^{\alpha-1}$, $(p_t, -p_{t+1})$ is the vector of partial derivatives of $\delta^t e_t^\eta(\theta k_t - k_{t+1})^\alpha$. Therefore, it holds that

$$\begin{aligned} & \delta^t e_t^\eta(\theta k_t - k_{t+1})^\alpha + p_{t+1} k_{t+1} - p_t k_t \\ & \geq \delta^t e_t^\eta(\theta x_t - x_{t+1})^\alpha + p_{t+1} x_{t+1} - p_t x_t \end{aligned}$$

for any feasible $\{x_t\}_{t \geq 0}$. Adding up this inequality over t , by (3.2) and (3.3),

$$\sum_{t=0}^{\infty} \delta^t e_t^\eta(\theta k_t - k_{t+1})^\alpha \geq \limsup_{T \rightarrow \infty} \sum_{t=0}^T \delta^t e_t^\eta(\theta x_t - x_{t+1})^\alpha$$

for any feasible $\{x_t\}$ such that $x_0 = k_0$. This implies that $\{k_t\}$ solves (2.8).

Q.E.D.

Corollary 1. *Suppose that a path $\{k_t\}$ satisfies (3.1), (3.2) and (3.3). Then $\{k_t\}$ is an equilibrium if and only if $e_t = k_t$ for $t = 0, 1, \dots$.*

Define $\lambda_t = k_{t+1}/k_t$. By the feasibility condition $(1 - \mu)k_t \leq k_{t+1} \leq \theta k_t$, it holds that $(1 - \mu) \leq \lambda_t \leq \theta$. Note that $\log \lambda_t$ is the growth rate of capital. By (2.6), the Euler equation associated with externality path $\{e_t\} = \{k_t\}$ can be transformed into a recursive system in a single variable, λ_t , as follows.

$$\lambda_{t+1} = \theta - (\delta\theta)^{\frac{1}{1-\alpha}} \cdot (\theta - \lambda_t)(\lambda_t)^{\frac{\eta}{1-\alpha}-1}. \quad (3.5)$$

Let $z_t = \theta - \lambda_t$ and $\beta = \frac{\eta}{1-\alpha} - 1$. It is more convenient to write the difference equation (3.5) in terms of z_t . Assume $\beta \geq 0$,⁴ and define a function $f : [0, \theta] \rightarrow R_+$ as

$$f(z) = (\delta\theta)^{\frac{1}{1-\alpha}} \cdot z(\theta - z)^\beta. \quad (3.6)$$

Then, the difference equation (3.5) can be expressed as

$$z_{t+1} = f(z_t). \quad (3.7)$$

The function f maps $[0, \theta]$ into the set of non-negative real numbers, R_+ . If $\beta = 0$, the graph of f is linear and positively sloped. If $\beta > 0$, then f is unimodal and satisfies $f(0) = f(\theta) = 0$. We focus on this case by assuming

$$\beta > 0. \quad (3.8)$$

The function f achieves its maximum at

$$\hat{z} = \frac{\theta}{\beta + 1}. \quad (3.9)$$

The maximum value is

$$f^{\max} = f(\hat{z}) = (\delta\theta)^{\frac{1}{1-\alpha}} \beta^\beta \left(\frac{\theta}{\beta + 1} \right)^{\beta+1}.$$

If this value does not exceed θ ,⁵ the function f maps the interval $[0, \theta]$ into itself. Also, note that

$$f'(z) = (\delta\theta)^{\frac{1}{1-\alpha}} (\theta - z)^{\beta-1} (\theta - (1 + \beta)z). \quad (3.10)$$

We guarantee that $f^{\max} < \theta$ and that $f'(0) > 1$ by assuming

⁴If $\beta < 0$, $f(0) = 0$ and, as $z \rightarrow \theta$, $f(z)$ monotonically increases to $+\infty$.

⁵See subsection 3.6 below for the case in which f^{\max} exceeds θ .

$$\textbf{Condition 1: } 1 < (\delta\theta)^{\frac{1}{1-\alpha}}\theta^\beta < \frac{(\beta+1)^{\beta+1}}{\beta^\beta}.$$

Let $\underline{\mu}$ be defined as $\underline{\mu} = \max\{0, 1 - (\theta - f^{\max})\}$. We choose μ from $(\underline{\mu}, 1]$ so that $f^{\max} < \theta - (1 - \mu)$.⁶ Let $J = [0, \theta - (1 - \mu)]$. f maps J into itself and (J, f) constitutes a dynamical system.

Lemma 2. *Suppose Condition 1 holds. A path $\{k_t\}$, $k_{t+1} = (\theta - z_t)k_t$, from $k_0 = \bar{k}$ and $z_t \in [0, \theta - (1 - \mu)]$, is an equilibrium if $z_{t+1} = f(z_t)$, $0 < z_t < \theta - (1 - \mu)$, $t = 0, 1, 2, \dots$, and $\sum_{t=0}^{\infty} \delta^t k_t^{\eta+\alpha} < \infty$.*

Proof: By Lemma 1, it suffices to demonstrate that $\{k_t\}$ satisfies the transversality and summability conditions. The value of the objective function is

$$\sum_{t=0}^{\infty} \delta^t e_t^\eta (\theta k_t - k_{t+1})^\alpha \leq \sum_{t=0}^{\infty} \delta^t k_t^\eta (\theta k_t)^\alpha = \theta^\alpha \sum_{t=0}^{\infty} \delta^t k_t^{\eta+\alpha}, \quad (3.11)$$

since $(1 - \mu)k_t \leq k_{t+1} \leq \theta k_t$. This implies that the summability condition is satisfied.

In order to demonstrate that the transversality condition is satisfied, it suffices to show that $\delta^t k_t^\eta (\theta k_t - k_{t+1})^{\alpha-1} k_{t+1} \rightarrow 0$. To this end, note that $z_t > \varepsilon$ for some $\varepsilon > 0$, since $z_0 > 0$, $f(f^{\max}) > 0$, and $f'(0) > 1$. Therefore,

$$\delta^t k_t^\eta (\theta k_t - k_{t+1})^{\alpha-1} k_{t+1} \leq \delta^t k_t^\eta (\theta k_t - (\theta - \varepsilon)k_t)^{\alpha-1} \theta k_t = \delta^t \theta \varepsilon^{\alpha-1} k_t^{\eta+\alpha}. \quad (3.12)$$

Since $\delta^t k_t^{\eta+\alpha}$ is summable, this implies that the transversality condition is satisfied.

Q.E.D.

3.2. Balanced Growth and Period Two Growth

We say that x is a period- n point of f if x is a fixed point of the n -th iterate of f , i.e., $x = f^n(x)$ but not a fixed point of the iterate of any order lower than n , i.e., $x \neq f^i(x)$ for $i = 1, 2, \dots, n - 1$. If x is a period- n point, we call the path $x, f(x), \dots, f^{n-1}(x)$, a period- n orbit or a period- n cycle. Clearly, if λ_t has a period- n cycle, our model economy will grow along a persistently oscillatory path. In

⁶See subsection 4.2 below for the case in which $0 < \mu < 1 - (\theta - f^{\max})$.

this section, we provide a sufficient condition under which the dynamical system (J, f) has a period-2 point and, therefore, the growth rate of k_t oscillates between a high and a low values.

Given $\beta > 0$, the dynamical system $f : J \rightarrow J$ has two fixed points. One is $z = 0$. The other is $z^* = \theta - (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}$. We call $\lambda^* = (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}$ a *balanced growth factor* and $\log \lambda^*$ a *balanced growth rate*.

Lemma 3. *The dynamical system $f : J \rightarrow J$ induces a unique balanced growth factor,*

$$\lambda^* = (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}. \quad (3.13)$$

Proof: Condition 1 guarantees the existence of a fixed point of f that is positive. **Q.E.D.**

Note that Condition 1 implies $\delta(\delta\theta)^{-(1+\frac{1}{(1-\alpha)\beta})} < 1$ which is equivalent to $\delta\lambda^{*\alpha+\eta} < 1$. Therefore, both the transversality and the summability conditions are satisfied on a balanced growth path. From the viewpoint of economics, of particular interest is the case in which $\log \lambda^*$ is positive so that the capital stock grows indefinitely when the economy is on the balanced growth path. Since $\lambda^* = \theta - z^* = (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}$, this condition is equivalent to

$$\textbf{Condition 2: } \theta < \delta^{-1}.$$

Persistent oscillations are possible when the steady state is locally unstable and the paths around the steady state fluctuate toward an attracting periodic orbit. By (3.10), local instability of the steady state can be obtained by strengthening the left-hand side of Condition 1,

$$\textbf{Condition 3: } 1 < \frac{(\beta + 2)^\beta}{\beta^\beta} < (\delta\theta)^{\frac{1}{1-\alpha}} \theta^\beta.$$

Conditions 1–3 force the graph of f to be unimodal over the segment $[0, \theta]$. To bound the invariant set of f (*i.e.*, the interval I such that $f(I) = I$) away from the origin, we assume that the image of the maximum value of f , $f(f^{\max})$, is smaller than the value of its pre-image, \hat{z} . This is equivalent to

$$\text{Condition 4: } (\delta\theta)^{\frac{2}{1-\alpha}} \left(\frac{\beta\theta}{\beta+1} \right)^\beta \left(\theta - (\delta\theta)^{\frac{1}{1-\alpha}} \left(\frac{\theta}{\beta+1} \right) \left(\frac{\beta\theta}{\beta+1} \right)^\beta \right)^\beta < 1.$$

Lemma 4. Let $I = [f(f^{\max}), f^{\max}]$. Then, under Conditions 1 through 4, f is smooth and onto I . Moreover, f also satisfies:

P1: At $\hat{z} = \frac{\theta}{1+\beta}$, $f'(\hat{z}) = 0$ and $f''(\hat{z}) < 0$. Furthermore $f'(z) > 0$ for $z \in (0, \hat{z})$ and $f'(z) < 0$ for $z \in (\hat{z}, \theta)$.

P2: $f(z) > z$ for $z \in (0, \hat{z})$.

Proof: Given $\beta > 0$, $f(z)$ is monotone increasing over $J' = [0, \hat{z}]$ and monotone decreasing over $J'' = [\hat{z}, f^{\max}]$. Moreover, since $f'(z^*) < -1$, it holds that, under Condition 4, $f(f^{\max}) < \hat{z} < f^{\max}$. Let $I' = [f(f^{\max}), \hat{z}] \subset J'$. Then, $f(I') = [f(f(f^{\max})), f^{\max}]$ and $f(I'') = [f(f^{\max}), f^{\max}]$. Since $f'(0) > 1$, and since $f(f^{\max}) < \hat{z} < z^*$ by $f'(z^*) < -1$, $f(f^{\max}) < f(f(f^{\max}))$. This implies $f(I'') \supset f(I')$, which implies $f(I) = I$. The remaining properties are straightforward. **Q.E.D.**

We should note that the equilibrium condition provided by Lemma 2 depends on the selected Euler path. In contrast, the next result provides a condition on the parameters of the model under which each Euler path is guaranteed to be an equilibrium path. Define

$$\hat{\theta} = \theta \left[1 - \frac{[(\delta\theta)^{\frac{1}{1-\alpha}} \theta^\beta]^2 \beta^\beta}{(1+\beta)^{\beta+1}} \left\{ 1 - \frac{(\delta\theta)^{\frac{1}{1-\alpha}} \theta^\beta \beta^\beta}{(1+\beta)^{\beta+1}} \right\}^\beta \right]. \quad (3.14)$$

Theorem 3.1. Suppose that Conditions 1 through 4 are satisfied. If $(\delta\hat{\theta})^{\frac{1}{1-\alpha}} (\hat{\theta})^\beta < 1$, any path $\{k_t\}$ that satisfies $k_{t+1} = (\theta - z_t)k_t$, $k_0 = \bar{k}$ and $z_{t+1} = f(z_t)$, $0 < z_t < \theta - (1 - \mu)$, is an equilibrium. Moreover, there is a z such that the growth rate of capital exhibits period-2 cycles.

Proof. We will first show that $\{k_t\}$ is an equilibrium path. By Lemma 4, there is a finite T such that $z_t > f^2(\frac{\theta}{1+\beta})$ for all $t > T$. Then, it holds that $\lambda_t = \theta - z_t < \hat{\theta} = \theta - f^2(\frac{\theta}{1+\beta})$ for $t > T$, from which $k_{t+1} = \lambda_t k_t < \hat{\theta} k_t$ for $t > T$. Hence,

$$\delta^t k_t^{1+(1-\alpha)\beta} < [\delta(\hat{\theta})^{1+(1-\alpha)\beta}]^{t-T} \delta^T k_T^{1+(1-\alpha)\beta}.$$

Therefore, $\sum_{t=0}^{\infty} \delta^t k_t^{1+(1-\alpha)\beta} < \infty$ if $\delta(\widehat{\theta})^{1+(1-\alpha)\beta} < 1$, i.e., $(\delta\widehat{\theta})^{\frac{1}{1-\alpha}}(\widehat{\theta})^\beta < 1$. Note that $0 < f^2(\frac{\theta}{1+\beta}) < \theta - \lambda_t$ for $t > T$. Let $\bar{\varepsilon} = f^2(\frac{\theta}{1+\beta})$. The transversality condition is given by $\lim_{t \rightarrow \infty} \delta^t k_t^\eta (\theta k_t - k_{t+1})^{\alpha-1} k_{t+1} = \lim_{t \rightarrow \infty} \delta^t k_t^{1+(1-\alpha)\beta} (\theta - \lambda_t)^{\alpha-1} \lambda_t \leq \lim_{t \rightarrow \infty} \delta^t k_t^{1+(1-\alpha)\beta} \bar{\varepsilon}^{\alpha-1} \theta = 0$, which holds, since $\delta^t k_t^{1+(1-\alpha)\beta}$ is summable. Thus, the first part of the theorem follows from the restriction upon $\widehat{\theta}$.

In order to prove the existence of period-2 cycles, take z_0 sufficiently close to 0. Then, along an Euler path from z_0 , $z_0 < f(z_0) < f^2(z_0)$, since $f(0) = 0$ and $f'(0) > 1$. Next, choose $z'_0 \in (z_0, z^*)$ sufficiently close to z^* . Since $f'(z^*) < -1$, $f^2(z'_0) < z'_0 < z^*$. In summary, it holds that

$$0 < z_0 < z'_0 < z^*, z_0 < f^2(z_0), \text{ and } f^2(z'_0) < z'_0.$$

Thus, by continuity of f^2 there is a fixed point $z'' = f^2(z'')$ which is a period-2 point for the dynamical system. **Q.E.D.**

3.3. Topological Chaos

The concept of chaos that we discuss first is due to Lie and Yorke (1975). We say that a dynamical system $h : X \rightarrow X$ is *topologically chaotic* if there is an uncountable subset S of X that contains no periodic points and that satisfies the following:

A. For every x and y in S such that $x \neq y$,

$$\limsup_n |h^n(x) - h^n(y)| > 0 \text{ and } \liminf_n |h^n(x) - h^n(y)| = 0.$$

B. For every x in S and every periodic point y in X ,

$$\limsup_n |h^n(x) - h^n(y)| > 0.$$

An uncountable subset S of X that satisfies the conditions **A** and **B** above is called a *scrambled set*.

Li and Yorke (1975) proves that if $h : X \rightarrow X$ is continuous and has a period-3 point, then the dynamical system (X, h) is topologically chaotic. On the other hand, it is clear that if h^n has a scrambled set, then h also has a scrambled set. These facts, together with Sarkovskii's theorem (1964), imply that a continuous map h that has a cycle of a period that is not a power of 2, is topologically chaotic.

The parameter restriction $(\delta\hat{\theta})^{\frac{1}{1-\alpha}}(\hat{\theta})^\beta < 1$, adopted in Theorem 3.1, is too strong to prove the existence of chaotic accumulation paths. It needs to be replaced with a weaker condition, one that does not uniformly restrict growth rates along every path. Along a chaotic path, the equilibrium growth factor λ_t exceeds the upper bound in Theorem 3.1 infinitely often, and it also becomes very small infinitely often. Consequently, we need to express the summability condition in terms of the growth factors λ_t and in such a way that the relative frequency with which λ_t visits small neighborhoods of $\theta - f(f^{\max})$ and $\theta - f^{\max}$, along any given path, can be explicitly taken into account. To this end, define

$$s_t = \log \delta + (\eta + \alpha) \log \lambda_t$$

and

$$S_\tau = \log \delta + \frac{1}{\tau}(\eta + \alpha) \sum_{t=0}^{\tau-1} \log \lambda_t = \frac{1}{\tau} \sum_{t=0}^{\tau-1} s_t.$$

Lemma 5. *Let $z_{t+1} = f(z_t)$ for $t = 0, 1, 2, \dots$ with $z_0 \in I$. Then $\lim_{\tau \rightarrow \infty} S_\tau$ exists. If, in particular, $\lim_{\tau \rightarrow \infty} S_\tau < 0$, then the path $\{k_t\}$, $k_{t+1} = (\theta - z_t)k_t$, is an equilibrium.*

Proof: Since the objective function can be transformed into

$$\sum_{t=0}^{\infty} \delta^t e_t^\eta (\theta k_t - k_{t+1})^\alpha = \sum_{t=0}^{\infty} \delta^t k_t^{\eta+\alpha} (\theta - \lambda_t)^\alpha,$$

in order to demonstrate that its value is finite, it suffices to show that $\delta^t k_t^{\eta+\alpha}$ is summable. To this end, note that

$$\begin{aligned} \delta^\tau k_\tau^{\eta+\alpha} &= \delta^\tau k_0^{\eta+\alpha} \left(\prod_{t=0}^{\tau-1} \lambda_t \right)^{\eta+\alpha} \\ &= k_0^{\eta+\alpha} e^{\log(\delta^\tau)} \left(\prod_{t=0}^{\tau-1} e^{(\eta+\alpha) \log \lambda_t} \right) \\ &= k_0^{\eta+\alpha} \exp \left\{ \sum_{t=0}^{\tau-1} (\log \delta + (\eta + \alpha) \log \lambda_t) \right\} \\ &= k_0^{\eta+\alpha} \exp \{ \tau S_\tau \}. \end{aligned}$$

Since $z_t \in I$, $0 < \theta - f^{\max} \leq \lambda_t \leq \theta - f(f^{\max})$. Therefore, there exist a and b , both finite, such that $a \leq s_t \leq b$ and that

$$\frac{1}{\tau} \left(\sum_{t=0}^{\tau-2} s_t + a \right) \leq S_\tau \leq \frac{1}{\tau} \left(\sum_{t=0}^{\tau-2} s_t + b \right).$$

Since

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \left(\left(\sum_{t=0}^{\tau-2} s_t + b \right) - \left(\sum_{t=0}^{\tau-2} s_t + a \right) \right) = \lim_{\tau \rightarrow \infty} \frac{b-a}{\tau} = 0,$$

S_τ has a limit as $\tau \rightarrow \infty$. If $S^* = \lim_{\tau \rightarrow \infty} S_\tau < 0$, then there exist a sufficiently large τ' and a sufficiently small $\nu > 0$ such that $S_\tau < S^* + \nu < 0$ for all $\tau > \tau'$. Hence,

$$\delta^\tau k_\tau^{\eta+\alpha} = k_0^{\eta+\alpha} \exp\{\tau S_\tau\} \leq k_0^{\eta+\alpha} \exp\{\tau(S^* + \nu)\}.$$

This implies that $\delta^\tau k_\tau^{\eta+\alpha}$ is summable. Let $\bar{\varepsilon} = f(f^{\max})$. The transversality condition is given by

$$\lim_{t \rightarrow \infty} \delta^t k_t^\eta (\theta k_t - k_{t+1})^{\alpha-1} k_{t+1} = \lim_{t \rightarrow \infty} \delta^t k_t^{\eta+\alpha} (\theta - \lambda_t)^{\alpha-1} \lambda_t \leq \lim_{t \rightarrow \infty} \delta^t k_t^{\eta+\alpha} \bar{\varepsilon}^{\alpha-1} \theta = 0,$$

which holds, since $\delta^\tau k_\tau^{\eta+\alpha}$ is summable. **Q.E.D.**

Example 1: Here, we fix parameters α, β and δ at $(\alpha, \beta, \delta) = (0.5, 5, 0.6)$ and allow the parameter θ to vary in the closed interval $[1.56, 1.66]$. To emphasize this fact, we write $f(z) = f(z; \theta)$. Note that this choice of parameters satisfies Conditions 1 through 4. It may be calculated that, as θ ranges over the interval $[1.56, 1.66]$, one has $-0.510826 < \lim_{\tau \rightarrow \infty} S_\tau < -0.385662$. In Figure 1, θ and $\lim_{\tau \rightarrow \infty} S_\tau$ are measured along the horizontal and vertical axes, respectively. We report the values of $\lim_{\tau \rightarrow \infty} S_\tau$ for a path beginning from the critical point of f . It is clear that $\lim_{\tau \rightarrow \infty} S_\tau$ remains negative. An average growth rate has been strictly positive for each path starting from the critical point $\theta/(1 + \beta)$.

Next we will find parameter values $(\alpha, \beta, \theta, \delta)$ at which f has a periodic path with a period not equal to a power of 2.

Example 2: Set $(\alpha, \beta, \delta, \theta) = (0.5, 5, 0.6, 1.599586)$. In Figure 2, the graphs of f^3 and f^5 are illustrated, respectively, by the upper and lower panels. As the figure indicates, f^3 has a single fixed point while f^5 has 11 fixed points. This implies that f does not have a cyclical orbit of period 3 (the only fixed point of

f^3 corresponding to the fixed point of f) but has a single steady state and two cyclical orbits of period 5. Since $(\alpha, \beta, \theta, \delta)$ satisfy Conditions 1 through 4, f has all the properties listed in Lemma 4 above. These facts imply that the dynamical system (I, f) is topologically chaotic and that the same is true for the equilibrium stock of capital defined as $k_{t+1} = (\theta - z_t)k_t$.

3.4. Existence of Ergodic Chaos

Consider a topological dynamical system generated by a continuous map $h : X \rightarrow X$, where X is a bounded closed interval. Let B_X be the set of all Lebesgue measurable subsets of X . Let ν be a measure defined on the measurable space (X, B_X) with $\nu(X) = 1$. (X, B_X, ν) constitutes a probability space. We say that the mapping h is *ergodic*, if the following two conditions are satisfied. (1) For any $A \in B_X$, $\nu(h^{-1}A) = \nu(A)$ (Stationarity). (2) For any $C \in B_X$ such that $h^{-1}C = C$, $\nu(C) = 0$ or 1 (Indecomposability). If h is ergodic, then ν is said to be an *ergodic invariant measure*.⁷ The dynamical system (X, h) might include multiple ergodic invariant measures. We are interested in a *well behaved* ergodic invariant measure such as *absolutely continuous with respect to the Lebesgue measure*. The dynamical system (X, h) is often said to be *ergodically chaotic* if it has a unique ergodic invariant measure that is absolutely continuous with respect to the Lebesgue measure.

This study adopts a concept of ergodic chaos based on the Schwartzian derivative. The Schwartzian derivative of $h : X \rightarrow X$, $Sh(z)$, is defined for $z \in X$ such that $h'(z) \neq 0$ as follows:

$$Sh(z) = \frac{h'''(z)}{h'(z)} - \frac{3}{2} \left(\frac{h''(z)}{h'(z)} \right)^2.$$

The following proposition is due to Misiurewicz (1980).

Proposition 1. *Let $a < b$ and $X = [a, b]$. Suppose that $h : X \rightarrow X$ satisfies the following conditions.*

1. h is of class C^3 .

⁷If h has an ergodic invariant measure ν , then for any ν -integrable function $g : X \rightarrow R$, $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} g(f^n(x)) = \int_X g(x)\nu(dx)$ holds for ν -almost all $x \in X$.

2. There is $c \in (a, b)$ such that $h'(c) = 0$ and $h''(c) < 0$. Furthermore, $a < x < c$ implies $h'(x) > 0$ and $c < x < b$ implies $h'(x) < 0$.
3. $h(x) > x$ for $x \in (a, c)$.
4. $Sh(x) < 0$ for $x \in (a, b)$, $x \neq c$.
5. There is an unstable periodic point $y \in X$ such that $y = h^n(c)$ for some $n \geq 2$.

Then, h has a unique ergodic invariant measure that is absolutely continuous with respect to the Lebesgue measure.

We have shown in Lemma 4 that f , when restricted to I , satisfies the properties 1–3 of Proposition 1. We shall now consider the properties 4 and 5. Given the form of f derived above, f' , f'' and f''' can be written as follows.

$$\begin{aligned} f'(z) &= (\delta\theta)^{\frac{1}{1-\alpha}}[\theta - (1 + \beta)z](\theta - z)^{\beta-1}; \\ f''(z) &= -(\delta\theta)^{\frac{1}{1-\alpha}}\beta[2\theta - (1 + \beta)z](\theta - z)^{\beta-2}; \\ f'''(z) &= (\delta\theta)^{\frac{1}{1-\alpha}}\beta(\beta - 1)[3\theta - (1 + \beta)z](\theta - z)^{\beta-3}. \end{aligned}$$

Hence, we have the following.

$$\begin{aligned} \frac{f'''(z)}{f'(z)} &= \frac{\beta(\beta - 1)[3\theta - (1 + \beta)z]}{[\theta - (1 + \beta)z](\theta - z)^2}, \\ \frac{f''(z)}{f'(z)} &= -\frac{\beta[2\theta - (1 + \beta)z]}{[\theta - (1 + \beta)z](\theta - z)}. \end{aligned}$$

By using these relationships, we may calculate the Schwartzian derivative

$$\begin{aligned} Sf(z) &= \frac{\beta[2(\beta - 1)(3\theta - (1 + \beta)z)(\theta - (1 + \beta)z) - 3\beta(2\theta - (1 + \beta)z)^2]}{2[\theta - (1 + \beta)z]^2(\theta - z)^2} \\ &= \frac{\beta[2(\beta - 1)(x + \theta)(x - \theta) - 3\beta x^2]}{2[\theta - (1 + \beta)z]^2(\theta - z)^2}, \end{aligned}$$

where $x = 2\theta - (1 + \beta)z$. Hence,

$$Sf(z) = -\frac{\beta[(\beta + 2)x^2 + 2(\beta - 1)\theta^2]}{2[\theta - (1 + \beta)z]^2(\theta - z)^2}.$$

Therefore, given $f'(z) \neq 0$, $Sf(z) < 0$ if the following condition is satisfied.

Condition 5: $\beta \geq 1$.

To the properties of f listed in Lemma 4 we can now add that, under Conditions 1–5:

P3. $Sf(z) < 0$ for $z \in I \setminus \{\hat{z}\}$.

We have,

Lemma 6. *Assume $\lim_{\tau \rightarrow \infty} S_\tau < 0$. Under Conditions 1–5, if there is $n \geq 2$ such that $f^n(\frac{\theta}{1+\beta}) = z^* = f(z^*)$, then, the equilibrium path $\{k_t\}$, $k_{t+1} = (\theta - z_t)k_t$, is ergodically chaotic.*

Proof: Under Conditions 1 through 5, the unique non-trivial steady state z^* is unstable. Therefore, $f : I \rightarrow I$ satisfies the properties 1–5 of Proposition 1 and it generates equilibrium paths that are ergodically chaotic. **Q.E.D.**

In the rest of this section, we exhibit parameter values at which our dynamical system generates equilibrium paths that are ergodically chaotic. By means of simulations we also compute some important features of such equilibria.

Example 3: As in Examples 1 and 2, fix parameters α, β and δ at $(\alpha, \beta, \delta) = (0.5, 5, 0.6)$ and allow parameter θ to vary in the closed interval $[1.56, 1.66]$. Note that this choice of parameters satisfies Conditions 1 through 5; therefore, the properties $P1$ through $P3$ hold. To apply Lemma 6, it suffices to check that $\lim_{\tau \rightarrow \infty} S_\tau < 0$ and $f^n(\frac{\theta}{1+\beta}; \theta) = z^*$ for some $n \geq 2$. As Example 1 demonstrates, for all $\theta \in [1.56, 1.66]$,

$$-0.510826 < \lim_{\tau \rightarrow \infty} S_\tau < -0.385662.$$

Hence, the first condition is satisfied. To check the second condition, write $g(\theta; n) = f^n(\frac{\theta}{1+\beta}; \theta)$ and $z^*(\theta) = \theta - (\delta\theta)^{\frac{-1}{\beta(1-\alpha)}}$. Figure 3.a reports values of $\theta \in [1.56, 1.66]$ along the horizontal axis and the corresponding values of $g(\theta; 20)$

and $z^*(\theta)$ along the vertical one. $g(\theta; 20) = z^*(\theta)$ for all those values of θ at which the two curves intersect.⁸ For any such θ , (I, f) is ergodically chaotic.

Figure 3.b illustrates the bifurcation diagram of $z_{t+1} = f(z_t; \theta)$. The critical point $\theta/(1 + \beta)$ is chosen as the initial condition in every round of simulations. For each value of $\theta = 1.56 + 0.0005m$, $m = 0, 1, 2, \dots$, we compute 2000 iterates of the critical point and then, corresponding to any such θ , we report in the graph the values of $f^n(\theta/(1 + \beta); \theta)$ for $n = 1001, \dots, 2000$.

For the same set of parameters one can also compute the values of θ at which the first few period-doubling bifurcations occur:

$$\theta_1 = 1.47150765\dots, \quad \theta_2 = 1.54550672\dots, \quad \theta_3 = 1.56246474\dots,$$

$$\theta_4 = 1.56612400\dots, \quad \theta_5 = 1.56690945\dots, \quad \theta_6 = 1.56707774\dots,$$

$$\theta_7 = 1.56711379\dots, \quad \theta_8 = 1.56712151\dots, \quad \theta_9 = 1.56712316\dots,$$

$$\theta_{10} = 1.56712351\dots, \quad \dots$$

Along such a cascade, an attracting cycle of period 2^n is created at θ_n from a cycle of period 2^{n-1} through a supercritical flip bifurcation. As it should be expected, in Figure 3.b the odd numbered cycles appear after the period-doubling process is completed, with a cycle of period three appearing last, as predicted by Sarkovskii's theorem.

A further, important regularity of our dynamical system is the following. Consider the following difference equations.

$$\begin{aligned} z_{t+1} &= (\delta\theta)^{1/(1-\alpha)} z_t (\theta - z_t)^\beta, \\ y_{t+1} &= Ay_t (1 - y_t)^\beta, \end{aligned}$$

⁸If $z^*(\theta) = g(\theta; n)$ for some n and θ , then $z^*(\theta) = g(\theta; N)$ for all $N \geq n$, because $z^*(\theta)$ is a fixed point of the map $f(\cdot; \theta)$. Therefore, the number of intersections of $z^*(\theta)$ and $g(\theta; n)$ monotonically increases, as n increases. For $n = 1$, there is no intersection. For $n = 3$, the intersection emerges for the first time. For $n = 6$, there exist 4 intersections. For $n = 8$, there exist 10 intersections. For $n = 10$, there exist 26 intersections. Figure 3.a depicts the case $n = 20$, where we have 58 intersections.

where $z_t = \theta y_t$, and $A = \delta^{1/(1-\alpha)} \theta^{(1/(1-\alpha))+\beta}$, so that the dynamical system in y_t is just a particular rescaling of the original dynamical system in z_t . Let $\alpha = 0.5$, $\beta = 5$, and $\delta = 0.6$.

Corresponding to the sequence of parameter values $\theta_1, \theta_2, \dots$ at which the period-doubling bifurcations occur, we obtain the sequence of A_n .

$$A_1 = 5.37824, \quad A_2 = 7.58226215\dots, \quad A_3 = 8.18415956\dots,$$

$$A_4 = 8.31927580\dots, \quad A_5 = 8.34852623\dots, \quad A_6 = 8.35480470\dots,$$

$$A_7 = 8.35615001\dots, \quad A_8 = 8.35643816\dots, \quad A_9 = 8.35649988\dots,$$

$$A_{10} = 8.35651310\dots, \quad \dots$$

It can be easily verified that the sequence of A_n satisfies the following universal relation

$$\lim_{n \rightarrow \infty} \frac{A_{n+1} - A_n}{A_{n+2} - A_{n+1}} = 4.669201609\dots,$$

where 4.669201609... is the universal constant, often referred to as ‘‘Feigenbaum Number’’. See Feigenbaum (1978, 1979). The sequence of A_n converges to a limit $A_\infty = 8.356516703\dots$ in a geometric progression as

$$A_n \approx A_\infty - c \cdot \mathcal{F}^{-n}$$

where \mathcal{F} is the Feigenbaum Number.

Finally, Figure 3.c depicts the magnitude of the Liapunov exponent at different values of θ .⁹ The Liapunov exponent is a measure of the complexity of the dynamical system. A positive value of the Liapunov exponent implies sensitive dependence on initial conditions. For the values of θ at which the two curves in Figure 3.a intersect, (I, f) is ergodically chaotic and the values of the Liapunov exponent in Figure 3.c are positive.

⁹The value of Liapunov exponent in Figure 3.c is the sample Liapunov exponent $\bar{\lambda}_T(z_0; \theta) = \frac{1}{T} \sum_{n=0}^{T-1} \log |f'(z_n; \theta)|$ from an initial value $z_0 = f^{N+1}(\frac{\theta}{1+\beta}; \theta)$ for sufficiently large N and T . We have used $N = T = 1000$. If $[(I, B_I, \nu), f]$ is ergodically chaotic and $\log |f'|$ is ν -integrable, then $\lim_{T \rightarrow \infty} \bar{\lambda}_T(z_0; \theta) = \int_I \log |f'(x; \theta)| \nu(dx)$ for ν -almost all $z_0 \in I$.

It is worth noticing that in correspondence to those values of $\theta \in [1.56, 1.66]$ at which at least one periodic point is stable, a window appears in the bifurcation diagram, Figure 3.b. For any such θ , the value of the Liapunov exponent is negative and the two curves in Figure 3.a do not intersect. This can be verified again by visual inspection of Figures 3.a and 3.c.

3.5. Robustness of Ergodic Chaos

In order to show that $f(\cdot; \theta)$ is ergodically chaotic, lemma 6 requires $f^n(\hat{z}; \theta)$ to hit an unstable steady state $z^*(\theta)$ for some $n \geq 2$. This implies that ergodic chaos may not be robust in the topological sense; if $f^n(\hat{z}; \theta) = z^*(\theta)$ for $n = n_0$ and $\theta = \theta_0$, it is not necessarily true that $f^n(\hat{z}; \theta)$ hits $z^*(\theta)$ for some $n \geq 2$ after a slight perturbation of θ from θ_0 . In what follows, however, we will show that ergodic chaos is robust in the measure theoretic sense.

Example 4: We fix α, β and δ at $(\alpha, \beta, \delta) = (0.5, 5, 0.6)$ and allow θ to vary in $[1.56, 1.66]$. Suppose that $f^n(\hat{z}; \theta)$ hits an unstable steady state $z^*(\theta)$ for $n = n_0$ and $\theta = \theta_0$. Now, consider the function $F(y) = y(1 - y)^\beta$ and the family of dynamical systems $F_A : y \rightarrow AF(y)$,

$$F_A(y) = Ay(1 - y)^\beta, \quad (3.15)$$

which is obtained from (3.6) by setting $y = z/\theta$ and $A = (\delta\theta)^{\frac{1}{1-\alpha}}\theta^\beta$.¹⁰ In order to apply Theorem B of Jakobson (1981) to this family of dynamical systems, note that $F_A : [0, 1] \rightarrow [0, 1]$ is a C^3 map, has a single nondegenerate critical point $1/(1 + \beta)$ and a steady state $y^* = 1 - A^{-\frac{1}{\beta}}$. It also has the negative Schwarzian derivative. Let $A_0 \equiv (\delta\theta_0)^{\frac{1}{1-\alpha}}\theta_0^\beta$ be such that there exists an n_0 for which $F_{A_0}^{n_0}(\frac{1}{1+\beta})$ hits the unstable steady state y^* . Under these conditions, Jakobson's theorem can be applied to establish that for any $\varepsilon > 0$, there exists $\rho > 0$ such that

$$m\{A \in \Lambda_A | A_0 \geq A \geq A_0 - \rho\} > \rho(1 - \varepsilon), \quad (3.16)$$

where Λ_A is the set of A for which $([0, 1], F_A)$ is ergodically chaotic and where m denotes the Lebesgue measure.¹¹ Therefore, for $\Lambda_\theta = \{(\delta^{-1}A^{1-\alpha})^{\frac{1}{1+\beta(1-\alpha)}} | A \in \Lambda_A\}$,

¹⁰We have used the same rescaled dynamical system $([0, 1], F_A)$ to obtain the Feigenbaum number.

¹¹See Remark XIII/5 in Jakobson (1981, p.87-88).

(3.16) implies that for any $\varepsilon > 0$, there exists $\rho > 0$ such that

$$m\{\theta \in \Lambda_\theta | \theta_0 \geq \theta \geq (\theta_0^{\frac{1+\beta(1-\alpha)}{1-\alpha}} - \delta^{\frac{-1}{1-\alpha}} \rho)^{\frac{1-\alpha}{1+\beta(1-\alpha)}}\} > \rho(1 - \varepsilon).$$

This implies that the ergodic chaos that f generates is robust with respect to a small perturbation in θ .

3.6. Cantor Set of Equilibria

For the sake of simplicity, we set $\mu = 1$ in the present subsection so that $J = [0, \theta]$. Consider the value of f at its critical point \hat{z} .

$$f(\hat{z}) = f^{\max} = (\delta\theta)^{\frac{1}{1-\alpha}} \beta^\beta \left(\frac{\theta}{\beta + 1} \right)^{\beta+1}.$$

So far we have restricted our analysis to parameter values for which $f^{\max} < \theta$ to insure that J is invariant under the action of f , i.e., $f(J) \subset J$. Our discussion has shown that for a set of parameter values with a positive measure, every initial condition k_0 can be associated to a continuum of equilibrium paths, most of which are ergodically chaotic.

For a given initial condition k_0 , the structure of the equilibrium set becomes even more complicated when $f^{\max} > \theta$ and J ceases to be invariant. In this case certain points $z_0 = \theta - \lambda_0$ leave the interval J , under the action of f , after a finite number of iterations: they are associated with inadmissible sequences of growth factors, along which $\lambda_T < 0$ for some T . These cannot be equilibrium sequences, as they obviously violate the feasibility condition.

We will now argue that, in these circumstances, the set of equilibria has the structure of a Cantor set, i.e. of a closed, totally disconnected and perfect subset of the bounded closed interval. Such a set has Lebesgue measure zero. Each one of its points corresponds to an equilibrium path starting from the same initial condition k_0 and displaying complex dynamics and sensitive dependence on initial conditions.

The intuition behind this statement builds upon results that are now standard in the theory of one-dimensional unimodal maps, and can be found, *e.g.*, in Devaney (1986) for the quadratic case. In our setting some extra complications arise because f is flat near $z = \theta$ and the technique adopted by Devaney (which relies on expansiveness everywhere) cannot be directly used.

Fix the value of θ and let the parameter $\beta > 1$, representing the ratio between the intensity of the external effect and the labor share, vary so that $f^{\max} > \theta$.

Denote with A_0 the set of points $z \in [0, \theta]$ at which $f(z) > \theta$. Then let $A_1 = \{z \in [0, \theta] \mid f(z) \in A_0\}$, and inductively define the sequence of intervals

$$A_n = \left\{ z \in [0, \theta] \mid f^n(z) \in A_0 \right\}$$

containing all the inadmissible initial conditions.¹² The equilibrium set reduces to

$$\mathcal{E} = [0, \theta] \setminus \left(\bigcup_{n=0}^{\infty} A_n \right). \quad (3.17)$$

It is straightforward to note that the procedure through which \mathcal{E} is constructed is reminiscent of the deletion algorithm generating the Cantor-Middle-Third set. The only difference here is that the open intervals we are removing are not symmetrically located around the middle point \hat{z} , due to the strong asymmetry of the map f .

This difference is, nevertheless, not essential and the proof that \mathcal{E} is a closed set may proceed in the standard way (see *e.g.* Devaney (1986)). It is less immediate to show that \mathcal{E} is disconnected and that it has Lebesgue-measure zero. The slope of f is zero at $z = \theta$. This makes it impossible to show directly that for $\epsilon > 0$ any non negligible interval of the type $[\theta - \epsilon, \theta]$ would contain points that are eventually mapped outside the interval $[0, \theta]$ by f^n . In spite of the fact that all intervals of that type are mapped into intervals of the type $[0, f(\theta - \epsilon)]$ and that the slope of f is very high near zero, the second iterate of f becomes as close to zero as one may please at values of $z \in [\theta - \epsilon, \theta]$ close enough to θ . Nevertheless one can still prove the following:

Theorem 3.2. *Under the assumptions of this subsection the equilibrium set \mathcal{E} defined above in equation (3.17) is a closed subset of the interval $[0, \theta]$, composed of an (uncountable) number of isolated points. The Lebesgue measure of \mathcal{E} is zero.*

Proof. The proof of the statement, which is rather lengthy, is contained in Lemmata A1-A3 in the Appendix. **Q.E.D.**

¹²We are referring here to initial conditions for the map f , i.e. to growth factors $\lambda_0 = k_1/k_0$ that cannot be chosen, in equilibrium, during the first (or, for that matter, any subsequent) period. Obviously the “true” initial condition for the economic system is k_0 , which is given and always admissible, whereas λ_0 is a non-predetermined control variable of the system.

4. Economic Properties of the Equilibria and Extensions

4.1. The Behavior of Prices and Quantities

So far, we have dealt with the growth factor $\lambda_t = \theta - z_t$. In this section we examine briefly the implied behavior of aggregate quantities and prices.

As we are interested in the behavior of the model when parameter values are such to generate chaotic behavior, some care should be placed in performing the simulation exercise. Suppose that $z^*(\theta_0) = g(\theta_0; n_0)$ for some n_0 and θ_0 . Although we do not know the exact numerical value of θ_0 , Example 4 demonstrates that if we substitute an approximate numerical value, slightly less than the true value of θ_0 , for θ , we can, with a positive probability, still reproduce ergodic chaos by means of numerical simulations. Consider one of the many parameter values at which $z^*(\theta)$ and $g(\theta; 8)$ intersect: $\theta_0 \approx 1.585770$. Limitations in the computer's numerical precision imply that our ability to calculate the exact value at which the intersection occurs is limited, i.e., the true value of θ_0 can only be approximated. In our simulations we used, as an actual value, the number $\tilde{\theta}_0 = 1.58576959774$. This parameter value satisfies $\tilde{\theta}_0 > 1.58576959774 > \theta_0 - \rho_0$, where $\rho_0 = 10^{-11}$. This is as good as an approximation to the true chaotic parameter value as it can get.

In each simulation, we use the critical point \hat{z} as an initial value, iterate $f(\cdot, \theta)$ N times, take the last T iterations,¹³ and substitute

$$\log k_t = \log k_{t-1} + \log(\theta - z_{t-1}) \quad (4.1)$$

and

$$\log c_t = [((1 - \alpha)\beta + 1) \log k_t + \alpha \log(z_t/b)] / (1 - \sigma) \quad (4.2)$$

for $(\log k_t, \log c_t)$ with $\sigma = 0.01$, $\mu = 1$ (i.e., $\theta = b$), $k_0 = 1$, and $t \geq 0$. From these values of k_t and c_t we compute the other sample paths using the national income account identities and the price formulas

$$q_t = b^{-\bar{\alpha}} \bar{\alpha} k_t^{\bar{\eta}} (\theta k_t - k_{t+1})^{\bar{\alpha}-1} \quad (4.3)$$

$$w_t = b^{-\bar{\alpha}} (1 - \bar{\alpha}) k_t^{\bar{\eta}} (\theta k_t - k_{t+1})^{\bar{\alpha}} \quad (4.4)$$

$$r_t = \bar{\alpha} b^{1-\bar{\alpha}} k_t^{\bar{\eta}} (\theta k_t - k_{t+1})^{\bar{\alpha}-1}, \quad (4.5)$$

¹³ $N = 12021$ for Figures 4.a-c, and $N = 12500$ for Figure 4.d. $T = 141$ for Figure 4.a, $T = 300$ for Figure 4.b, and $T = 1000$ for Figures 4.c-d. Figures 4.a-c plot the last 100 samples of the T iterations.

where $\bar{\alpha} = \alpha/(1 - \sigma)$ and $\bar{\eta} = \eta/(1 - \sigma)$.

The behaviors of the most important macroeconomic variables are depicted in Figures 4.a-d. Figure 4.a reports the sample paths of capital stock $\log k_t$, and consumption $\log c_t$ while Figure 4.b reports the associated sample paths for the prices of investment q_t , and labor w_t , and for the rental rate of capital r_t , all expressed in units of the consumption good. In Figure 4.c we have reported the sample paths of two different measures of aggregate output, $\log y_t$ and $\log \tilde{y}_t$. These two indices are defined, respectively, as $y_t = c_t + q_t k_{t+1}$ and $\tilde{y}_t = c_t + q_{t_0} k_{t+1}$. The latter corresponds to the National Income Accounting convention of using the relative prices of some base year t_0 in computing aggregate output over subsequent periods. We set $t_0 = 75$. Finally, Figure 4.d reports the sample distributions of the growth factors of the stock of capital ($\lambda_t = k_{t+1}/k_t$). Given our choice of parameter values the balanced growth value of the growth rate is $\log \lambda^* = 0.0195843$ and along the sample path we have computed we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^T \log \lambda_t = 0.0199023$$

which is remarkably close and confirms the essential symmetry of the asymptotic distribution of growth rates generated by the underlying chaotic process.

As expected from $((1 - \alpha)\beta + 1)/(1 - \sigma) > 1$, c_t grows faster than k_t . As expected from our choice of θ , the sample paths of all prices and quantities exhibit chaotic fluctuations. Some of these fluctuations are remarkably wide and, from an empirical point of view, not very appealing. This is true for our measures of consumption and of the real wage. On the other hand the fluctuations exhibited by the stock of capital, its relative price and the rental rate are much less incredible. While it is certainly not our intention to claim “realism” for our simple model economy, we find the dynamic behavior of this second set of quantities and prices not completely unrealistic. Finally, we should stress the remarkable difference in the behavior of the two measures of aggregate output. Indeed the time-profile of $\log \tilde{y}_t$, which corresponds to the index adopted in National Income Accounting data, is extremely realistic, exhibiting recurrent but apparently random fluctuations of a relatively small size around a smooth growth trend.

The criteria adopted in these simulations for the choice of parameter values should also be briefly commented. Recall that the parameter δ represents the discount factor of future utilities; we may assume that the closer δ is to 1, the shorter the length of one period is. In contrast, the parameter β represents the

strength of the externality assumed in the model. In order to obtain a fully developed bifurcation diagram like that shown in Figure 3.b, the values of these two parameters cannot be chosen independently. In particular, in order to increase the value of δ , we must increase that of β . This implies that in order to observe clear chaotic dynamics with a larger discount factor, we have to deal with an economy with a stronger external effect. For $\delta = 0.75$, β should be greater than 9. For $\beta = 1$, δ should be less than 0.25. The parameter α does not appear, at least in our extensive numerical simulations, to affect the pattern of bifurcation significantly. We used the parameter values $(\alpha, \beta, \delta) = (0.5, 5, 0.6)$ for Figures 3 and 4. The unit of time adopted is therefore quite longer than a year and more of the order of about a five year period.

If we are to obtain period-2 cycles, we can reduce β for a given δ , or we can increase δ for a given β , without violating neither the transversality condition nor the positive growth condition as shown below.

Example 5: For $(\alpha, \beta, \delta) = (0.5, 3.1, 0.6)$, the fixed point $z^*(\theta)$ loses its stability and undergoes a supercritical flip bifurcation, as θ increases and crosses 1.6535614... . Figure 5 depicts sample paths of the logarithms of k_t and c_t for $(\alpha, \beta, \delta, \theta) = (0.5, 3.1, 0.6, 1.66)$, which demonstrate the existence of period 2 cycles for the case of a smaller value of β than those discussed above ($\beta = 5$), *i.e.*, for the case of a weaker external effect. For Figure 5, we use again the critical point $\theta/(1 + \beta)$ as an initial value, iterate $f(\cdot; \theta)$ 5000 times, take the last 300 iterations, substitute (4.1) and (4.2) for $(\log k_t, \log c_t)$ with $\sigma = 0.01$, $\mu = 1$ (*i.e.*, $\theta = b$), $k_0 = 1$, and $t \geq 0$, and plot the last 100 samples of $(\log k_t, \log c_t)$.

As expected from $((1 - \alpha)\beta + 1)/(1 - \sigma) > 1$, c_t grows faster than k_t . As expected from our choice of $(\alpha, \beta, \delta, \theta)$, the sample paths of $\log k_t$ and $\log c_t$ are subject to period-2 cycles. In this case the balanced growth value of the growth rate of capital is $\log \lambda^* = 0.00290762$ while along a sample path we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^T \log \lambda_t = 0.00258582.$$

4.2. Less Than Full Depreciation

Earlier on we introduced the simplifying assumption according to which a rate of capital depreciation μ is sufficiently close or equal to one such as $1 - (\theta - f^{\max}) < \mu \leq 1$. $\mu = 1$ corresponds to full depreciation. This simplified the algebra required to characterize the basic properties of the equilibrium set. We should try to relax

this assumption and consider the consequences of setting $0 < \mu < 1 - (\theta - f^{\max})$. In general this implies that the map f looks as in Figure 6. Only a subinterval $[0, \phi] = \tilde{J} \subset [0, \theta]$, $\phi < \theta$, is now invariant under the action of f .

Visual inspection suggests that for values of μ close to one the behavior of the equilibrium trajectories remain pretty much the same as that discussed in earlier sections. As μ decreases, unfortunately the structure of the equilibrium set becomes harder to characterize. We will try nevertheless to provide here a short and heuristic discussion.

Denote with Π the set of preimages of ϕ under finite iterations of the map f . In other words, Π is defined as the set of all those $z \in [0, \phi]$ for which there exists an $n > 0$ such that $f^n(z) = \phi$. In particular denote with \mathcal{P} the set of direct preimages of ϕ , i.e., all those $z \in [0, \phi]$ such that $f(z) = \phi$. The reader will notice the geometric analogy between this case and the one we just considered for $f^{\max} > \theta$, an analogy worth exploiting to shorten the discussion. Define

$$\Lambda = [0, \phi] \setminus \Pi.$$

It is clear that most points in $[0, \phi]$ will belong to Π but contrary to the previous case one cannot claim this set to be of Lebesgue measure zero.

When the point ϕ is mapped into Π then all the equilibria with an initial condition $z_0 \in \Pi$ are eventually periodic of some finite period n . In the special case in which $f(\phi) \in \mathcal{P}$, there exists a continuum of attracting period two cycles. Notice that the periodicity of this cycle is independent of the initial condition $z_0 \in \Pi$.

For the special case in which Λ contains an aperiodic orbit (which will be true for example whenever the hypotheses of Proposition 1 are satisfied) then one could try to apply again Theorem B of Nusse [1987] to conclude that the set of points in $[0, \phi]$ which do not converge to the periodic point of period p is of zero Lebesgue measure.¹⁴

On the other hand, when $f(\phi) \notin \Pi$ the asymptotic behavior of the equilibrium growth rate will depend on the specific structure of Λ . As we have already mentioned, one cannot show that in general Λ has measure zero even if it obviously has a Cantor-like structure. Whether the asymptotic behavior over Λ will be periodic or aperiodic can be decided by applying here the same criteria we developed in section 3.

¹⁴The map we are now considering is not three times differentiable at two isolated points. It is a tedious but straightforward matter to verify that this does not affect the proof.

5. Conclusions

We have presented a class of two-sector models in which unbounded accumulation is triggered by the existence of constant returns in the production of capital good and persistent oscillations are brought about by the presence of increasing returns in the production of the consumption good. The increasing returns are due to the positive external effect generated by the aggregate stock of capital.

The model predicts that when the external effect is very strong, the growth rate will oscillate forever along cycles of long periodicity or even over a chaotic attractor. We have provided a full characterization of the conditions under which this occurs and illustrated the features of these endogenous growth cycles both analytically and by means of numerical simulations.

We have shown that a particularly dramatic form of indeterminacy of equilibrium may arise in such a simple economy. For a given initial state of the system, that is: given the aggregate stock of capital at time $t = 0$, there exists an infinite set of equilibria. In certain cases this set of equilibria has positive Lebesgue measure and can be parameterized by the whole interval $(0, \theta)$ of growth rates which are feasible from k_0 . In other cases this set of equilibria is still composed of an uncountable number of points but it has the structure of a Cantor set and zero Lebesgue measure.

In both circumstances we have shown that there exists parameter values at which indeterminacy is “global”. With global indeterminacy we refer to the case in which *any* pair of equilibria departing from the same initial condition k_0 , stay away from each other even in the infinitely far future and any equilibrium path different from the balanced growth path never grow at the balanced growth rate even in the long-run. These two features of the equilibria distinguish the theoretical properties of our model from those of other models of indeterminacy due to external effects (*e.g.* Benhabib and Farmer (1994), Benhabib and Perli (1994) and Boldrin and Rustichini (1994)). In the latter class of models, indeterminacy of equilibria is obtained for paths starting in a neighborhood of the balanced growth path and asymptotically converging to the balanced growth path.

Our results demonstrate that in models with positive externalities in which unbounded accumulation is possible, trend and cycles in aggregate variables can be simultaneously generated by the same endogenous economic mechanism. We believe this line of investigation to be worthy of further consideration.

Appendix Extend the function f over the whole real line.

$$\tilde{f}(z) = \begin{cases} -(z - \theta)^2 & z > \theta \\ f(z) & z \in [0, \theta] \\ f'''(0)\frac{z^3}{3} + f''(0)\frac{z^2}{2} + f'(0)z & z < 0 \end{cases}$$

Then $\tilde{f} : \mathfrak{R} \rightarrow \mathfrak{R}$ is C^3 , with $S\tilde{f} \leq 0$ and $S\tilde{f} < 0$ if $z \neq \theta$, $\hat{z} = \theta/(1 + \beta)$.

Lemma A1 *There exists a $\bar{\beta}$ such that for all $\beta > \bar{\beta}$ the map \tilde{f} has a cycle of period three. Also, $f^{\max} < \theta$ at values of β near enough $\bar{\beta}$.*

Proof. Consider the behavior of the third iterate \tilde{f}^3 . Denote with $LM = \tilde{f}^{-2}(\hat{z})$ the left pre-preimage of the critical point and with $Lm = \tilde{f}^{-1}(\hat{z})$ the left preimage of the critical point. Then $\tilde{f}^3(LM) = f^{\max}$ and $\tilde{f}^3(Lm) = \tilde{f}(f^{\max})$. So $0 < LM < Lm < z^* < \theta$. It is easy to see that $f^{\max} \rightarrow \infty$ as $\beta \rightarrow \infty$. This implies that $\tilde{f}^3(LM) \rightarrow \infty$ and $\tilde{f}^3(Lm) \rightarrow -\infty$ monotonically with β . Since \tilde{f} is continuous and $LM, Lm \in (0, \theta)$, there will exist a $\bar{\beta}$ such that for all $\beta > \bar{\beta}$ there exists a \tilde{z} satisfying

$$\tilde{f}^3(\tilde{z}) = \tilde{z}, \quad \text{and} \quad LM < \tilde{z} < Lm.$$

Since $\tilde{f}(\tilde{z}) \neq \tilde{z}$, this proves the existence of a period three. That $f^{\max} < \theta$ is trivial. **Q.E.D.**

The next step is to show that every stable periodic orbit of the map \tilde{f} will attract the critical point \hat{z} . While tedious, this amounts to verifying that the proof given in Singer (1978) can be replicated for a map defined over the whole real line and with zero Schwartzian derivative at a finite number of isolated points. To save space we will follow the seven steps of the proof as given in Collet and Eckmann (1980, pp. 97-100) and check that they are all satisfied also here.

Lemma A2 *For all $\beta > 1$ every stable periodic orbit of \tilde{f} attracts the critical point \hat{z} .*

Proof. We will proceed by checking that all the seven properties derived in Collet and Eckmann (1980, pp. 97-100) are satisfied also here.

1) is just a property of Schwartzian derivatives, i.e. if f and g are C^3 then $S(f \circ g)(x) = (Sf)(g(x))g'(x)^2 + Sg(x)$.

2) Let $\mathcal{P}^n(\theta)$ be the set of preimages of θ under \tilde{f}^n . Then one can verify by direct computation that for all $n \geq 1$, $S\tilde{f}^n < 0$ everywhere on \mathfrak{R} but at $\mathcal{B} = \{\hat{z}, \theta, 0\} \cup \mathcal{P}^1(\theta)$.

3) For all $z \in \{\mathfrak{R} \setminus \mathcal{B}\}$, $|f'|$ has no positive local minimum.

4) The map \tilde{f} has finitely many points of period n for every integer $n \geq 1$. To see this, let $\text{Per}(\tilde{f})$ be the set of periodic points. Then it is clear that $\mathcal{P}^n(\theta) \cap \text{Per}(\tilde{f}) = \emptyset$. Suppose now that for some n , $\tilde{f}^n(x) = x$ for infinitely many x . Then by the mean value theorem we must have $g'(x_k) = (\tilde{f}^n)'(x_k) = 1$ for infinitely many $x_k \notin \mathcal{P}^n(\theta)$. Moreover, since the cardinality of $\mathcal{P}^n(\theta)$ is bounded above by 2^n , for infinitely many k 's $[x_k, x_{k+1}] \cap \mathcal{P}^n(\theta) = \emptyset$ will hold. Point 3) above implies that $|g'|$ must vanish on those intervals. But this contradicts the fact that f , and hence \tilde{f} and g have finitely many critical points.

5) If $a < b < c$ are consecutive fixed points of $g = \tilde{f}^n$ and if $[a, c]$ contains no critical points of g , then $g'(b) > 1$.

6) Let $z \in \mathfrak{R}$ be a stable fixed point for $g = \tilde{f}^n$, and assume $|g'(z)| < 1$. Then the *stable manifold* of z is the set of points converging to z under iteration of g and the *semilocal stable manifold* of z is the connected component of the stable manifold of z , which contains z . Then by our definition of \tilde{f} , (r, s) is the only possible form that the semilocal stable manifold can assume, where r and s are two finite numbers which are either both fixed points of g , or a period-2 cycle for g or one a fixed point of g and the other its preimage under g . Mimicking Collet and Eckmann (1980, p. 99) one can check that in all cases g must have a critical point in $[r, s]$ which is attracted to z .

7) Finally consider the case in which $g = \tilde{f}^n$ has a fixed point $g(z) = z$ at which $|g'(z)| = 1$. Only the situation in which $-\infty < z < \infty$ needs to be considered. By 4) there must be a neighborhood (a, b) of z containing no other fixed points of g . Again by the same arguments of Collet and Eckmann (1980, p. 100), $g(y) > y$ must hold for $y \in (a, z)$. Let d be the minimum value of y for which $g(y) \geq y$. Observe first that $(d, z) \cap \mathcal{P}^n(\theta) = \emptyset$ since by definition $x \in (d, z)$ implies $g^k(x) \rightarrow z$ as $k \rightarrow \infty$ and $x \in \mathcal{P}^n(\theta)$ implies $g^k(x) \rightarrow 0$ as $k \rightarrow \infty$. Then either $d = -\infty$ or $g(d) = d$ must hold. The first case contradicts the fact that $\tilde{f}(y) \rightarrow -\infty$ when $y \rightarrow \mp\infty$. In the second case (d, z) must contain a critical point of \tilde{f} as proved in Collet and Eckmann (1980, p. 100). **Q.E.D.**

Lemma A3 *Under the assumptions of this Appendix the equilibrium set \mathcal{E} has Lebesgue measure zero.*

Proof. Formally we want to show that the set of points \mathcal{E} such that $z \in \{\mathfrak{R} \setminus \mathcal{E}\} \iff \{\tilde{f}^n(z) \rightarrow -\infty\}$ has Lebesgue measure zero when $f^{\max} > \theta$. We will make use of the following theorem proved in Nusse (1987).

Theorem. Assume that $f : X \rightarrow X$ is C^3 over the non-trivial interval X and it satisfies the following hypotheses:

- there exists at least one aperiodic point for f ;
- $Sf(x) \leq 0$ for all x such that $f'(x) \neq 0$;
- the set of points originating orbits that do not converge to an absorbing boundary point of X is a non-empty compact set;
- the orbits of each critical point of f converge to either some asymptotically stable periodic point or to an absorbing boundary point of X ;
- the fixed points of f^2 are isolated.

Then:

- 1. the set of points whose orbits do not converge to an asymptotically stable periodic point or to an absorbing boundary point of X has Lebesgue measure zero;
- 2. there exists a positive integer p such that almost every point in X is asymptotically periodic with (not necessarily primitive) period p , provided that f is bounded.

We have shown (Lemma A1) that \tilde{f} has a period-3 cycle for admissible values of β . This implies the existence of an aperiodic point. \tilde{f} also satisfies the second hypothesis by construction. The third one is also satisfied as the set of points in question is the intersection of a collection of closed and bounded subsets of the real line. We have proved in Lemma A2 that the critical point \hat{z} converges to $-\infty$ under repeated iterations of \tilde{f} . The last hypothesis is obviously true.

Given that the trajectory of the critical point \hat{z} converges to $-\infty$ there are no asymptotically stable periodic orbits for f because if such an orbit existed it would attract the trajectory starting at the critical point. Hence Nusse's theorem implies our statement. **Q.E.D.**

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