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ACYCLICITY AND STABILITY FOR INTERTEMPORAL OPTIMIZATION MODELS*

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1. INTRODUCTION

It has long been recognized that no simple and general criterion exists for assuring the global stability of dynamical systems which are solution of intertemporal optimization problems. The literature on the argument is quite large and we refer to McKenzie [8] for a general discussion. In other places (see [3] and [4]), we have shown that, under general conditions, it is possible to find values of the discount factor such that the optimal program shows chaotic behaviors. In this paper, we tackle the opposite problem and we show that the notion of acyclicity of a binary relation over a compact set can be useful to obtain global stability results which are independent of the level of the discount factor. The results that are achieved in this way can be interpreted, we believe, as the appropriate n-dimensional generalization of the monotonicity results obtained by Dechert-Nishimura [6] and Benhabib-Nishimura [2] in the one-dimensional case. Moreover, the techniques adopted enable us to do our proofs without differentiability assumptions.

The notion of acyclicity has been widely used in economic theory but, as far as we know, it has never been applied to Optimal Growth Models. A paper by Hammond on the dynamic behavior of an adaptive consumer, see [7], has been the stimulus for the present research and it is close in spirit to our approach.

The paper proceeds as follows: Section 2 contains the main mathematical definitions and the central stability theorem for acyclic policy functions, then in Section 3 we show how a binary relation can be associated with a dynamic programming model. In Section 3, we provide also a criterion to detect the presence of acyclicity from the properties of the short run return function.

Section 4 is devoted to a similar purpose for unidimensional dynamical systems; in this case, making use of a famous theorem of Sarkovskii [12], we are able to provide a complete characterization of acyclic optimal programming problems in terms of the structure of the return function. Here we also show the relation between our method and the results on the monotonicity of the one-dimensional model quoted above.

We end the paper with some brief conclusions and suggestions for future research.

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2. ACYCLIC PATHS ARE STABLE

Our starting point will be the introduction of the mathematical problem in all its generality.

Let X be a compact subset of R^n and P a binary relation over X; i.e. P is a subset of $X \times X$. We will write yPx for $(x, y) \in P$. In the sequel we will use P to denote both the binary relation and the subset of $X \times X$ defined as $P = \{(x, y) \in X \times X, yPx\}$.

An obvious economic interpretation is: the action y is preferred to the action x. Notice however that the assumptions of completeness and transitivity are not required in this framework.

DEFINITION 1. *A binary relation P over X is called cyclic if there exists a sequence $(x_1, \ldots, x_n)$ of distinct points in X such that $x_2 P x_1$, $x_3 P x_2, \ldots, x_n P x_{n-1}$ and $x_1 P x_n$. P is said to be acyclic if it is not cyclic.*

We remind that the transitive completion P^* of P is defined as the set $P^* = \{(x, y) \in X \times X$, s.t. there exists a finite sequence $(x_1, \ldots, x_n)$ of points in X such that $x_1 Px$, $x_2 P x_1, \ldots, x_n P x_{n-1}$, $yPx_n\}$, then the above definition simply states that P is acyclic if and only if P^* is irreflexive.

Now consider a discrete-time dynamical system defined by a continuous map f: $X \to X$ where X is a compact subset of R^n. We indicate with $\Omega(f)$ the set of non-wandering points of f which are defined in the following way (see [9] page 118):

DEFINITION 2. *A point $p \in X$ is a wandering point for f if there exists a neighborhood V of p and a positive integer t_0 such that $f^t(V) \cap V = \emptyset$, for $|t| > t_0$. Otherwise we say that p is a non-wandering point. The collection of all such p's is the non-wandering set $\Omega(f)$.*

A classical discussion of the structure of the set Ω for discrete and continuous dynamical systems is in Smale [13]. An informal characterization of Ω is suggested here in terms of "recurrence." In fact Ω contains not only equilibria and periodic orbits, but also all the other points of X which are not transported "too far" from themselves under the iterations of f: Ω is where the action is.

The two definitions above are all we need to state and prove the following:

THEOREM 1. *Let f: $X \to X$ be a continuous map on the compact space X and P a binary relation over X. Assume that f and P satisfy the following conditions:*

(*i*) *$f(x)Px$ for all $x \in X$ such that $f(x) \neq x$.*
(*ii*) *P is open and acyclic.*

Then we have:

$$\Omega(f) = Fix(f),$$

where $Fix(f)$ is the set of all the fixed points of f.

PROOF. P open means that the set $P = \{(x, y) \in X \times X, yPx\}$ is an open subset of $X \times X$. Then also P^* is open and we have the following fact: being P acyclic, $x_n P^* y_n$ implies $y_n \not{P}^* x_n$ where $\not{P}^*$ defines the complement of P^* in $X \times X$. For $x_n \to x$, $y_n \to y$ as $n \to \infty$ the closedness of $\not{P}^*$ implies $y \not{P}^* x$.

Now let us assume that a point $\bar{x}$ exists such that $\bar{x} \in \Omega(f)$ and $\bar{x} \notin Fix(f)$. Since $\bar{x} \in \Omega(f)$ we can take two sequences $x_k \to \bar{x}$ and $y_k \to \bar{x}$ as $k \to \infty$ and a sequence of integers $n_k \to \infty$ such that: $f^{n_k}(x_k) = y_k$ where f^{n_k} indicates the usual n_k-th iterate of f. Now: $x \notin Fix(f)$ permits to assume $f^r(x_k) \neq f^{r+1}(x_k)$ for all $0 \leq r \leq n_k$, in fact otherwise $y_k = f^{n_k}(x_k)$ would be a fixed point of f and, by continuity, $\bar{x} = \lim y_k$ would be a fixed point too in contrast to our hypothesis.

Then we can write $f^{n_k}(x_k) P^* f(x_k)$ and, taking the limits for $k \to +\infty$ we obtain $f(\bar{x}) \not{P}^* \bar{x}$. But this contradicts the hypothesis that $f(\bar{x}) \neq \bar{x}$.

Thus, $\Omega(f) \subset Fix(f)$ must hold. The reverse is immediate and so the Theorem is proved. Q.E.D.

With respect to the above result, the following facts are worth being stressed. Dynamical systems satisfying the assumption of Theorem 1 turn out to possess "trivial dynamics," in particular whenever it is possible to show that $Fix(f)$ is a singleton then global stability is assured because all trajectories converge toward the unique fixed point.

The sense in which we use the word "trivial" should be clarified. It refers to the fact that, in general, $\Omega(f)$ contains many things, as for example the α- and ω-limit sets of the system and the periodic points, etc. (for details, see [9] and [13]). Some of these objects can have very complicated structures: for example the so-called "strange attractors" belong to the non-wandering set.

The reduction of $\Omega(f)$ to the set of fixed points of f indicates a high degree of simplicity and regularity of the motion. From this fact comes the definition of trivial dynamics.

3. APPLICATION TO OPTIMAL PATHS

Consider the standard economic programming problem:

$$W_\delta(x_0) = \operatorname{Max} \sum_{t=0}^{\infty} u(x_t, x_{t+1}) \delta^t$$

$$\text{s.t. } x_{t+1} \in \Gamma(x_t) \tag{1}$$

$$x_0 \text{ a fixed value in } X$$

where:

A1) X is a convex and compact subset of R^n.

A2) $u: X \times X \to R$ is continuous, concave and strictly concave in the second argument.

A3) $\Gamma: X \to X$ is a continuous and compact valued correspondence, with convex graph in $X \times X$ and such that $x \in \Gamma(x)$ for all $x \in X$.

A4) The discount factor $\delta \in (0, 1)$.

Using the value function $W_\delta(x_0)$, it is quite natural to associate a binary relation to (1). We remind that the value function satisfies the following functional equation:

$$W_\delta(x_0) = \text{Max } \{u(x_0, x_1) + \delta W_\delta(x_1), \quad \text{s.t. } x_1 \in \Gamma(x_0)\} \tag{2}$$

It is also well known that W_δ turns out to be continuous and strictly concave under our hypothesis.

DEFINITION 3. *Given problem (1) define the binary relation P_δ over X as follows:*

$$yP_\delta x \quad \text{iff} \quad [u(x, x) + \delta W_\delta(x)] < [u(x, y) + \delta W_\delta(y)].$$

Using Definition 3, the application of Theorem 1 to problems of the form (1) is straightforward. At this purpose, let $(x_0, x_1, x_2, \ldots)$ be the optimal solution to (1) when x_0 is the given initial value. Then it is possible to define a map τ_δ: $X \to X$ associating x_t to x_{t+1} for every $t = 0, 1, 2, \ldots$, i.e., τ_δ is defined implicitly as $x_{t+1} = \tau_\delta(x_t)$. A more precise discussion of the properties of the policy function τ_δ and its application to optimal accumulation problems is contained in [3] and [4].

By the Bellman's Principle, τ_δ is such that:

$$\tau_\delta(x) = \text{Arg Max } [u(x, y) + \delta W_\delta(y)] \quad \text{s.t. } y \in \Gamma(x) \tag{3}$$

Clearly τ_δ satisfies all the assumptions of Theorem 1 if P_δ of Definition 3 is acyclic. Unfortunately it is very difficult to study the acyclicity of P_δ for the general case due to the presence of the unknown element $W_\delta(y)$ in (3). It is obvious that any direct attempt to evaluate whether the relation of Definition 3 produces cycles of any order, results in an impossible task as soon as the simplest nonlinear case is considered. It would therefore be useful to possess indirect methods able to detect the acyclicity of P_δ from the one-period return function $u(x, y)$. Along this line of research, a very first result is provided in the following theorem.

THEOREM 2. *Let $V(x, y)$ be a continuous map from R^{2n} in R and Ψ and Φ be continuous from R^n in R. The class of functions*

$$U(x, y) = V(x, y) + \Psi(x) + \Phi(y)$$

induces an acyclic relation P_δ for every $\delta \in [0, 1)$, if and only if V satisfies the following inequality:

$$\sum_{t=1}^{n} V(x_t, x_t) \geq \sum_{t=1}^{n} V(x_t, x_{t+1}) \tag{4}$$

for every sequence $(x_1, \ldots, x_n)$ with $x_{n+1} = x_1$.

Proof. Sufficiency will be proved by contradiction. Assume that (4) is satisfied but there exists a n-cycle $(x_1, \ldots, x_n)$ for $U(x, y)$. In this case, we must have:

$$\Psi(x_t) + \Phi(x_t) + V(x_t, x_t) < \Psi(x_t) + \Phi(x_{t+1}) + V(x_t, x_{t+1})$$

for $t = 1, \ldots, n$ with $x_{n+1} = x_1$. This follows from repeated applications of Definition 3. Summing up from 1 to n we get:

$$\sum_{t=1}^{n} V(x_t, x_t) < \sum_{t=1}^{n} V(x_t, x_{t+1}),$$

a contradiction.

Let us now show necessity. Assume (4) is not satisfied for a certain sequence $\{x_1, \ldots, x_n\}$, we must have:

$$(*) \qquad \sum_{t=1}^{n} V(x_t, x_t) < \sum_{t=1}^{n} V(x_t, x_{t+1})$$

We will show that there exists a continuous function Φ from R^n in R such that $\{x_1, \ldots, x_n\}$ is a cycle for $U(x, y) = V(x, y) + \Psi(x) + \Phi(y)$ with Ψ generic. At this purpose set:

$$\beta_t = V(x_t, x_t) - V(x_t, x_{t+1}), \qquad t = 1, \ldots, n-1$$

It is easy to see that a continuous function Φ can be constructed in such a way that:

$$(**) \qquad \beta_t < \Phi(x_{t+1}) - \Phi(x_t) < \beta_t + \varepsilon$$

$t = 1, \ldots, n-1$, for an appropriate $\varepsilon > 0$. By construction, the left side of (**) implies: $x_2 P x_1, x_3 P x_2, \ldots, x_n P x_{n-1}$, where P is the binary relation associated to $V(x, y) + \Phi(y)$. Summing up over t the right side of (**) and substituting back from the definition of β_t we get:

$$\Phi(x_n) - \Phi(x_1) < \sum_{t=1}^{n-1} [V(x_t, x_t) - V(x_t, x_{t+1})] + (n-1)\varepsilon$$

from which:

$$\Phi(x_n) - \Phi(x_1) + [V(x_n, x_n) - V(x_n, x_1)] < \sum_{t=1}^{n} [V(x_t, x_t) - V(x_t, x_{t+1})] + (n-1)\varepsilon$$

By an appropriate choice of ε (*) implies that the right side of the latter inequality can be made strictly negative. Then:

$$V(x_n, x_n) + \Phi(x_n) < V(x_n, x_1) + \Phi(x_1)$$

so that $x_1 P x_n$ and $\{x_1, \ldots, x_n\}$ is a cycle for $V(x, y) + \Phi(y)$. Q.E.D.

The following proposition characterizes a class of return functions that satisfies Theorem 2 and that are widely used in economic theory.

PROPOSITION 1. *Assume $V: R^n \to R$ is concave. If $u(x, y) = \Psi(x) + \Phi(y) + V(y - x)$ is used in problem (1) then the relation P_δ given in Definition 3 is acyclic for every value of the discount rate δ in the interval* [0, 1).

PROOF. It is immediate. We know that P_δ acyclic means that $u(x, y) + \delta W_\delta(y) = \Psi(x) + \Phi(y) + V(y - x) + \delta W_\delta(y) = \Psi(x) + \Phi^*(y) + V(y - x)$ is acyclic too. Then we have only to prove that any concave functional $V(y - x)$ satisfies condition (4) of Theorem 2 to conclude that P_δ is acyclic. Let $(x_1, \ldots, x_n)$ be any sequence in X, we have to show that:

$$nV(0) \geq \sum_{t=1}^{n} V(x_{t+1} - x_t), \qquad (\text{set } x_{n+1} = x_1),$$

that is:

$$V(0) \geq (1/n) \cdot \sum_{t=1}^{n} V(x_{t+1} - x_t)$$

which is immediately verified using the concavity of V, i.e.:

$$V(0) = V\left[(1/n) \sum_{t=1}^{n} (x_{t+1} - x_t)\right] \geq (1/n) \cdot \sum_{t=1}^{n} V(x_{t+1} - x_t) \qquad \text{Q.E.D.}$$

Notice that also in this case Ψ and Φ are generic functions. We used this result in Boldrin and Montrucchio [5] to obtain a stability theorem for the capital accumulation paths of a competitive firm facing convex adjustment costs.

4. ACYCLICITY AND MONOTONICITY FOR ONE-DIMENSIONAL MAPS

In this section we restrict our attention to one-dimensional problems. This will enable us to study the relation between the notion of acyclicity and the monotonicity properties proved in [2] and [6] for the one-dimensional case. Moreover, we will show that in this case acyclic return functions can be characterized in a very simple way.

The result is achieved through the application of a powerful theorem due to Sarkovskii [12] on the ranking of the cycles of a continuous map over an interval of the real line. Let us assume the following regularity conditions for the binary relation $P \subset X \times X$:

(i) P is open as a subset of $X \times X$.
(ii) The set $L(x) = \{y \in X: yPx\}$ is convex for every $x \in X$.
(iii) If, for a particular point $x' \in X$, $L(x') = \emptyset$, then for any sequence $x_n \to x'$ such that $L(x_n) \neq \emptyset$ there exists a sequence $y_n \to x'$ with $y_n \in L(x_n)$.

We can prove:

THEOREM 3. *Let X be an interval on the real line and let $P \subset X \times X$ satisfy conditions (i) through (iii) above, then P is acyclic if and only if P is antisymmetric, i.e. if xPy implies $y \not P x$.*

PROOF. See Appendix.

The Theorem has been given in a general form but it can be easily specialized to the utility maximizing case we have been considering above.

COROLLARY 1. *Let the objective function $U(x, y)$ be continuous and strictly quasi-concave and set yPx if and only if $U(x, x) < U(x, y)$. Then P satisfies the regularity conditions (i) through (iii) above and Theorem 3 can be applied.*

PROOF. See Appendix.

The interpretation of the above results in terms of optimal paths converging to a stable optimal stationary state should be fairly clear. We complete the analysis of the one-dimensional case providing a characterization of the return functions which are acyclic. By Theorem 3 we need only to exclude cycles of period two. Using Theorem 2 we can state:

PROPOSITION 2. *Let X be an interval on the real line. The objective functions $U(x, y) = V(x, y) + \Psi(x) + \Phi(y)$ with V, Ψ and Φ continuous are acyclic if and only if:*

$$V(x, x) + V(y, y) \geq V(x, y) + V(y, x)$$

for all pairs $(x, y) \in X \times X$.

Finally, the relation between acyclic and monotonically increasing policy functions is given in the following:

PROPOSITION 3. *Consider a problem of the type (1) assuming $\dim X = 1$ and u differentiable. If $u_2(x, y)$ is increasing in x for every given y then $u(x, y)$ satisfies Proposition 2. In particular for u twice differentiable, this is true if $u_{21} \geq 0$.*

PROOF. By assumptions we can write $[u_2(x_1, y) - u_2(x_2, y)] \geq 0$ when $x_1 \geq x_2$ for every y in X. Integrating the above inequality we have:

$$\int_{x_2}^{x_1} [u_2(x_1, y) - u_2(x_2, y)]\, dy \geq 0$$

from which

$$u(x_1, x_1) - u(x_1, x_2) \geq u(x_2, x_1) - u(x_2, x_2)$$

and then:

$$u(x_1, x_1) + u(x_2, x_2) \geq u(x_2, x_1) + (x_1, x_2) \qquad \text{Q.E.D.}$$

5. CONCLUSIONS

We believe that this work represents a first attempt to apply the notion of acyclicity to the stability of infinite-horizon optimal growth models. The research is still quite open and, with some notable exceptions, most of the results presented here are only partial and not fully satisfactory.

A problem, in particular, seems to be critical for future improvements: it concerns the characterization of the class of short-run return functions u that give rise to acyclic relations. Theorem 2 is a good starting point, but it is not very easily applicable to high dimensional maps. The case considered in Proposition 1 is certainly important, but not exhaustive. The characterization provided in Theorem 3 for the one-dimensional case does not seem to be replicable in a higher dimensional setting. A possible conjecture is that by applications of the notion of α-concavity some further insights could be gained.

So much for the "pars destruens": we see also some positive contributions in our approach. A concise list can be as follows:

(a) The stability property holds independently of the value of δ, this in contrast to a large part of the existing literature in which the Turnpike is proved to be attractive only for large values of the discount parameter.
(b) Our treatment of the problem is completely based upon continuity and does not require any smoothness assumption. This is a useful application of Occam's Razor.

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APPENDIX

To prove Theorem 3 we need the following Lemma:

LEMMA. *Assume the binary relation $P \subset X \times X$ satisfies the regularity conditions (i) through (iii) given in Section 4. Then for every sequence of distinct points $\{x_1, \ldots, x_n\} \in X$, such that $x_{i+1}Px_i$ $(n+1=1)$ it is possible to find a continuous map $\mu: X \to X$ with the following properties:*

(a) $\mu(x_i) = x_{i+1}$
(b) $\mu(x)Px$ for every $x \neq \mu(x)$.

PROOF. As we have seen the correspondence $L(x)$ on X can be empty valued. For our purposes we will construct a new correspondence $L^*(x)$ with convex and nonempty values. Take a sequence $\{x_1, \ldots, x_n\} \in X$ as given, then define:

$$L^*(x) = \begin{cases} L(x) & \text{if} \quad L(x) \neq \varnothing \quad \text{and} \quad x \notin \{x_1, \ldots, x_n\}. \\ \{x_{i+1}\} & \text{if} \quad x = x_i, \quad \text{some} \quad i \in \{1, \ldots, n\}. \\ \{x\} & \text{if} \quad L(x) = \varnothing. \end{cases}$$

Using (i) and (iii) it is immediately seen that $L^*(x)$ is lower-hemicontinuous. Then for the Michael's Selection Theorem (see for example [1], page 82) there exists a continuous selector $\mu: X \to X$ over $L^*(x)$ (i.e. such that $\mu(x) \in L^*(x)$) which satisfies by construction the properties (a) and (b). Q.E.D.

PROOF OF THEOREM 3. Assume P is antisymmetric and suppose the existence of an n-cycle $\{x_1, \ldots, x_n\}$ such that $x_{i+1} P x_i$ $(n + 1 = 1)$.

The existence of a map $\mu: X \to X$ such that $\mu(x_i) = x_{i+1}$ and $\mu(x)Px$ follows from the Lemma above. But then we can apply Sarkovskii's Theorem (see [12]). In fact $\{x_1, \ldots, x_n\}$ is a periodic orbit with period $n \geq 2$ for the continuous map μ and this implies the existence of a period-two cycle, i.e. a couple of points $\{y_1, y_2\}$ such that $y_1 = \mu(y_2)$ and $y_2 = \mu(y_1)$, so that $y_1 P y_2$ and $y_2 P y_1$ contradicting the assumption.

The "only if" part of the Theorem follows straightforwardly from this same reasoning. Q.E.D.

PROOF OF COROLLARY 1. To prove that (i) and (ii) hold is trivial, (iii) can be shown as follows. For the particular P we are considering $L(x') = \varnothing$ for some $x' \in X$ means $U(x', y) \leq U(x', x')$ for all y in X. Now take $x_n \to x'$ and $L(x_n) \neq \varnothing$, this means that there exists another sequence y_n such that $U(x_n, x_n) < U(x_n, y_n)$. This, together with the continuity of U, implies that for every convergent subsequence $y_n \to y$ we have: $U(x', x') \leq U(x', y)$, which gives $U(x', x') = U(x', y)$ under the hypothesis $L(x') = \varnothing$. But $U(x', \cdot)$ is strictly quasi-concave so that $x' = y$ must hold. Q.E.D.

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